Lec 10/24

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Find all triples (a,b,c) of integers for which $\exists S \subset \mathbb{Z}$ with

$$(S+a) \sqcup (S+b) \sqcup (S+c) = \mathbb{Z}. \quad (*)$$

If $\exists S$, how many S?

Prob: if S satisfies $(*)$ for some (a,b,c) then S is periodic i.e. $\exists k>0$ s.t. $S = S-k$

Prove/disprove: S is a finite union of progressions

Ex: Let P be a partition-regular property s.t. if A has P and $n \in \mathbb{N}$ then nA has P.
Call a set P^* if it intersects any P set.
Show that a P^* set is multiplicatively thick.

Is there a notion of largeness responsible for partition regularity of $x+y=y$

Jake's Theorem: there are actually uncountably many notions of largeness which do the job.

(take notion to be: containing sub-IP of a given IP).

Ex: There are uncountably many different IP sets in $\mathbb{N}$

! Open: is it true that $\forall$ finite partition $\mathbb{N} = \dot{\bigcup}_{i=1}^{r} C_i$ one C_i contains $\{x, y, x+y, x \cdot y\}$????!!!!

Dyn. Form of VdW

$S = \{ n \in \mathbb{Z} : \|n\alpha\| < \varepsilon \}$ ($\|\alpha\|$ = distance to closest integer)

Ex: Claim: S is IP^* (actually, S is Δ^*)

Theorem $\forall \alpha \notin \mathbb{Q}$, $\{n\alpha \bmod 1 : n \in \mathbb{N}\}$ is dense in $[0,1]$.

Pf $[0,1]$ is compact so $\exists n_i \nearrow \infty$ s.t. $n_i \cdot \alpha \longrightarrow x_0 \in [0,1]$.
then $\exists i < j$ $\|(n_i - n_j)\alpha\| < \varepsilon$ □

$x \xrightarrow{T} x+\alpha$ is a "nice" transformation on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

$T^n(0) = n\alpha \bmod 1$ poincare recurrence principle.

generalize to $(n\alpha, n\beta) \in \mathbb{T}^2$

what about $n^2\alpha$??

it is not Δ^* but it is IP^*.

The following (call it TVdW) is equiv. to vdW:

$\forall$ compact metric space (X, d) and any homeo $T : X \longrightarrow X$, any $k \in \mathbb{N}$, any $\varepsilon > 0$,

$\exists\, n \in \mathbb{N}$ and $x \in X$ s.t. $\operatorname{diam}(\{x, T^n x, \ldots, T^{kn} x\}) < \varepsilon$

Prove: vdW $\iff$ TvdW hint $\Leftarrow$: use symbolic space (of colorings)

it's easier to prove for top minimal system, whence you get main result.

hint: Fact: any top. dyn. system (X, T) contains (homeo T)

minimal subsystem: $Y \subset X$ (closed), $TY \subseteq Y$,

and Y doesn't contain a proper subsystem.

Example: $X = \mathbb{T}$, $T : x \longmapsto x + \frac{1}{5}$

(X,T) is not minimal. it contains many minimal subsystems $\{x, x+\frac{1}{5}, x+\frac{2}{5}, x+\frac{3}{5}, x+\frac{4}{5}\}$

Another one: $X = \mathbb{T}^2$, $T : (x,y) \longmapsto (x+\alpha, y+\alpha)$

X is not minimal, $\{x=y\}$ is a subsystem

and this is a minimal subsystem

($n\alpha$ is dense)

Ex: (X,T) is minimal iff any point has dense orbit

$x \longmapsto x^2$ in $[0,1]$.