Lec 10/22

Monday, October 22, 2018   14:23

1) a) An IP $\cap$(Syndetic$-t$) is IP for some t (in this case, IP is a set containing FS).

d) $\forall x$, I is IP iff I/x is IP

e) $\forall x$, A is IP* iff A/x is IP*.

EX: Prove AP-richness of arbitrary Syndetic set is $\approx$ VdW's Theorem

Ex: If $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, one C_i contains a finite sum set & a finite product set?

Ex: Is there a set of positive natural density which contains no shift of an IP-set?
Yes! maybe square-free (but maybe not). Some arbitrarily close to 1.

1892:
Hilbert's Irreducibility Theorem: If $f(x,y) \in \mathbb{Z}[X,Y]$ and is irreducible, then $\exists a \in \mathbb{Z}$ s.t. $f(a,Y)$ is also irreducible.

Hilbert's cube lemma: $Q(x_0; \overbrace{x_1, \dots, x_k}^{\text{distinct}}) = x_0 + \left\{ \sum_{i=1}^{k} \varepsilon_i x_i : \varepsilon_i = 0 \text{ or } 1 \right\}$.

$\forall k \in \mathbb{N}$, $\forall$ finite partition $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, one C_i contains ∞-many shifts of a certain k-cube.

Ex: Let $A \subset \mathbb{N}$, $d(A) = \frac{1}{2}$ and $\forall t \in \mathbb{N}$, $d(A \cap (A-t)) = \frac{1}{4}$.
is it true that $\forall 0 < t_1 < t_2 < \cdots < t_n \in \mathbb{N}$, $d(A \cap (A-t_1) \cap \cdots \cap (A-t_n)) = \frac{1}{2^{n+1}}$?

Pf ~~if IP + t is in one C_i, we are done~~

Claim: If $A \subset \mathbb{N}$, $d(A) > 0$, then A satisfies the conclusion of the lemma.

Pf $\exists n_1$ s.t. $\bar{d}(A \cap (A - n_1)) > 0$

Exercise
Hint: $\bar{d}(A) > \frac{1}{10}$, L long enough,
$\frac{A}{L} > \frac{1}{10}$ so $\frac{A-t}{L} > \frac{1}{10}$ $\forall t = 1, 2, \dots, 10$
so some $A - t_1$ intersects $A - t_2$ with density $> \frac{1}{100}$, etc.

So $\exists n_2 > n_1$ s.t. $\bar{d}((A \cap (A-n_1)) \cap ((A \cap A - n_1) - n_1)) > 0$

So $\exists n_2 > n_1$ s.t. $\bar{d}\big((A \cap (A - n_1)) \cap ((A \cap (A - n_1)) - n_1)\big) > 0$

but this is $A \cap (A - n_1) \cap (A - n_2) \cap (A - (n_1 + n_2))$.

that's the end of the proof

so some $A - t_1$ intersects $A - t_2$ with density $> \frac{1}{100}$, etc.

December Reading: Khintchine: Three pearls of Number Theory

Try to see how this claim (and Hilbert's Cube Lemma) generalize to other semigroups like $V_{\mathbb{F}_p}$.

Try to transfer graph statements to 0-1 matrix statements.

Reading Start Ch 14

Ex: $\{0,1\}^{\mathbb{N}}$ is homeomorphic to $\{0, 1, \dots, r-1\}^{\mathbb{N}}$

$\Omega = \{0, 1, \dots, r-1\}^{\mathbb{Z}}$

any $\omega \in \Omega$ corresponds to a r-coloring of $\mathbb{Z}$.

Let $\sigma : \Omega \longrightarrow \Omega$ be defined as $\sigma(\omega)(n) = \omega(n+1)$ $\forall n \in \mathbb{Z}$.

Let $X_\omega = \overline{\{\sigma^n(\omega) : n \in \mathbb{Z}\}} \subseteq \Omega$. Then (X_ω, σ) is a "nice" topological system

orbital closure

$\sigma : X_\omega \longrightarrow X_\omega$