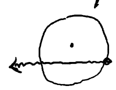
Lec 4/20

Friday, April 20, 2018 10:18

<u>Recall</u> $f(z) = \operatorname{Log} z$. taylor expand around $z_0 = e^{i3\pi/4}$

taylor series converges thru cut.



$f(z) = -i\pi + \operatorname{Log}(e^{i\pi} z)$ (rotating branch cut).

"parking lot domain"

$f(z) = \log|z| + i\theta$ where $-\infty < \theta < \infty$.

Now define $\log\left(\prod_{j=1}^{n}(z-z_j)\right) = \ln\left(\prod_{j=1}^{n}|z-z_j|\right) + i\sum_{j=1}^{n}\underbrace{\arg(z-z_j)}_{\in\,(-\infty,\,\infty)}$

$\log\left((z-z_1)^{m_1}(z-z_2)^{m_2} f_1(z)\right)$ so $f_1(z)$ is never 0.

$= m_1 \log(z-z_2) + m_2 \log(z-z_2) + \log(f_1(z))$

function no longer on $\mathbb{C}$ but instead on some Riemann Surface.

<u>Topics</u>

quite elementary

- Complex #s & properties:
  - polar rep. euler formula. finding roots of polys. exponentials & logs.
- Complex function: $f(\underset{z}{x+iy}) = u(x,y) + i\,v(x,y)$. $f: U \longrightarrow \mathbb{C}$.
- Plane topology: open & closed sets. boundaries, closures, interior pts, seqs, convergence, acc. pts.
- Continuity: uniform continuity, sequence of fns, compact sets,

3: Analyticity: complex derivative vs diffability in the real sense.

real

$f(z) = u(x,y) + i\,v(x,y)$. if $u, v \in C^1(\mathbb{R}^2)$ then f is diffable in real sense

$\partial_z = \frac{1}{2}(\partial_x - i\,\partial_y)$. $\partial_{\bar z} = \frac{1}{2}(\partial_x + i\,\partial_y)$

$f(z) = f(z_0) + a(z-z_0) + b(\bar z - \bar z_0) + E(z)$, where $|E(z)|$ =

$f(z) = f(z_0) + a(z-z_0) + b(\bar{z}-\bar{z}_0) + E(z)$ where $\frac{|E(z)|}{|z-z_0|} \to 0$ as $z \to z_0$.

f is complex diffable if $b = 0$.

Complex derivative (↓ same) exists in nhd of z_0 iff $\frac{\partial}{\partial \bar{z}} f = 0$

→ f analytic at z_0 if f has $\mathbb{C}$-derivative in a nhd of z_0.

Complex derivative at a point vs in a nhd.

$f(z) = |z|^2$ has complex derivative at $z=0$: $\lim_{h \to 0} \frac{f(h) - f(0)}{h} = 0$.

however $f(z)$ has no complex derivative in any nhd of 0.

Analytic fn $\Longleftrightarrow$ Cauchy Riemann + continuity of 1st partial derivatives of u & v.

$u_x = v_y$, $u_y = -v_x$

$$f' = u_x + i v_x = u_x - i u_y = v_y + i v_x = v_y - i u_y$$

Ref & Imf

are harmonic (since f has any # of derivatives, proved later).

Some harmonic fns do not have conjugates, but they all do in a simply connected domain.

D simply connected iff all u harmonic have conjugate.

exponential & trig functions.

branches of inverse:

$$\text{Arctan}\, z = \frac{1}{2i} \text{Log}\left(\frac{1+iz}{1-iz}\right) \quad \text{on } D =$$

branches of $z^\lambda = \exp(\lambda \log z)$.

When $\lambda = \frac{1}{n}$, $n \in \mathbb{Z}^+$ we get n distinct branches.

Complex integral:

paths γ. reverse path $-\gamma$. sum of paths $\gamma_1 + \gamma_2$.

$\gamma : [a,b] \longrightarrow \mathbb{C}$

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$

independent of parameterization of γ.

$\|$

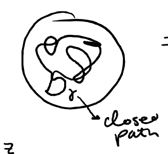
$\int_{|\gamma|} f(z)\,dz$ is well-defined

another kind is $\int_\gamma f(z)\,|dz| = \int_a^b f(\gamma(t))\,|\gamma'(t)|\,dt$ arc length integral.

Propn: $\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz$. $\int_{-\gamma} f(z)\,|dz| = \int_\gamma f(z)\,|dz|$.

f has a primitive F i.e. $F'(z) = f(z) \Longrightarrow \int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$.

Local Cauchy Theorem:

f analytic in a disk: 

$\Rightarrow \int_\gamma f(z)\,dz = 0$.

Winding # $n(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}$

Local Cauchy integral: $2\pi i\, n(\gamma, z_0)\, f(z_0) = \int_\gamma \frac{f(z)\,dz}{z - z_0}$.