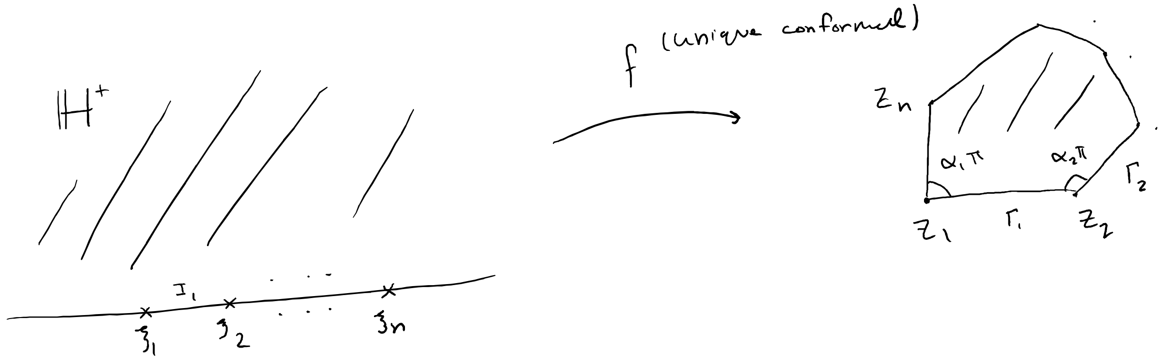
Lec 4/18

Wednesday, April 18, 2018 10:18

Review session weds 6 – 7:30 MW 154.



$-\infty < \xi_1 < \xi_2 < \cdots < \xi_n \leq \infty$

Theorem:

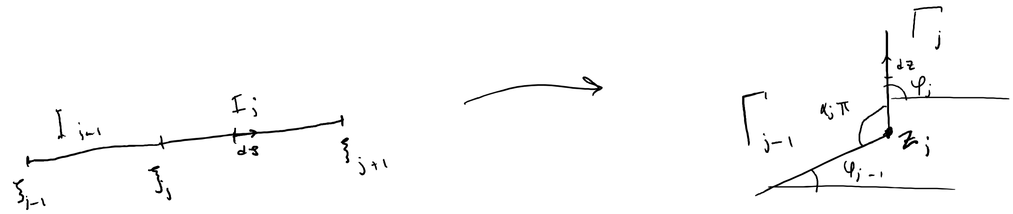
$$H(\zeta) = \begin{cases} \dfrac{f'(\zeta)}{\prod_{j=1}^{n} (\zeta - \xi_j)^{\alpha_j - 1}} & \text{when } \xi_n \neq \infty \\[2em] \dfrac{f'(\zeta)}{\prod_{j=1}^{n-1} (\zeta - \xi_j)^{\alpha_j - 1}} & \text{when } \xi = \infty \end{cases}$$

$H(\zeta)$ is constant.

Proof Last time: Showed H to be analytic at $\zeta = \xi_j$ for finite ξ_j.

We know $H(\zeta)$ is analytic in $\mathbb{H}^+ \cup \{I_j\}_{j=1}^{n}$.

So H is analytic in $\mathbb{H}^+ \cup \mathbb{R}$.



for $\zeta \in I_j$, $z \in \Gamma_j$.

$$\operatorname{Arg}\left(\frac{dz}{d\zeta}\right) = \operatorname{Arg}(f'(\zeta)) = \operatorname{Arg}(dz) - \operatorname{Arg}(d\zeta) = \varphi_j$$

for $\zeta \in I_{j-1}$, $\operatorname{Arg}(f'(\zeta)) = \underbrace{\varphi_{j-1} = \varphi_j - (1-\alpha_j)\pi}_{\text{geometric.}}$

In case $\zeta_n \neq \infty$, for $\zeta \in I_j$

$$\operatorname{Arg}\left(\prod_{k=1}^{n} (\zeta-\zeta_k)^{\alpha_k - 1}\right) = \sum_{k=1}^{n} \operatorname{Arg}\left((\zeta-\zeta_k)^{\alpha_k - 1}\right) = \sum_{k=j+1} (\alpha_k - 1)\pi$$

on I_j, $\operatorname{Arg}(H(\zeta)) = \varphi_j - \sum_{k=j+1} \pi(\alpha_j - 1)$

on I_{j-1}, $\operatorname{Arg}(H(\zeta)) = \varphi_{j-1} - \sum_{k=j} \pi(\alpha_j - 1) = \varphi_j + (\alpha_j - 1)\pi - \sum_{k=j} \pi(\alpha_j - 1) = \operatorname{Arg}(H(\zeta))$ on I_j.

$\operatorname{Arg}(H(\zeta))$ is continuous so it's constant on $\mathbb{R}$, say θ_0.

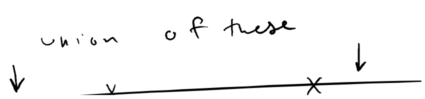
Argument is valid even when $\zeta_n = \infty$

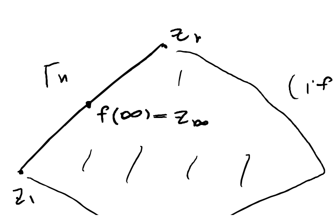
So $e^{-i\theta_0} H(\zeta) \in \mathbb{R}^+$ and $e^{-i\theta_0} H(\zeta)$ is analytic in $\mathbb{H}^+$ and cts in $\mathbb{H}^+ \cup \mathbb{R}$.

so Schwarz reflection principle implies $e^{-i\theta_0} H$ (and so H) is entire.

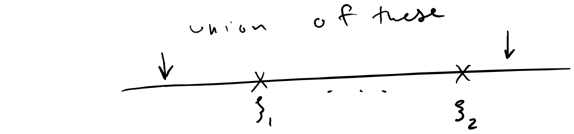
If $\zeta_n \neq \infty$,

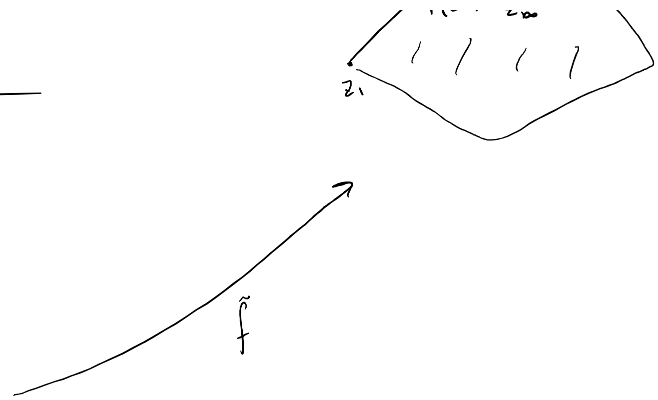
union of these



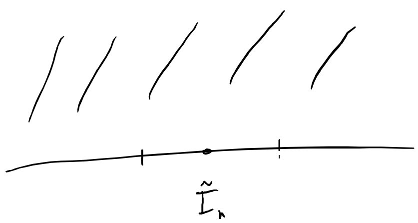


(if $\zeta_n = \infty$, $z_\infty = z_n$)





introduce $\chi = \frac{-1}{\xi}$.



It follows that $\tilde{f}(\chi)$ is analytic around $\chi = 0$.

$$\lim_{\xi \to \infty} \xi^2 f'(\xi) = -\tilde{f}'(0).$$

eg Consider meromorphic fn

~~eg~~ $\dfrac{\pi^2}{\sin^2 \pi z}$ want to write in infinite sum.

at $z = z_n = n$, we have poles.

Principal part at z_n, $\dfrac{1}{(z-n)^2}$.

Consider the fn $E(z) = \dfrac{\pi^2}{\sin^2 \pi z} - \displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$

take $z \in K$ compact and contained in $B(\rho/2, 0)$.