Lec 3/20

Tuesday, March 20, 2018 10:30

i) conversely: assume $\lim_{z \to z_0} |f(z)| = \infty$. Consider the fn $h(z) = \frac{1}{f(z)}$.

$\exists\, \Delta^*(z_0, \delta)$ s.t. h is analytic & $\lim_{z \to z_0} h(z) = 0$. So h analytic in $\Delta(z_0, \delta)$

Since h is not identically 0, $h(z) = (z - z_0)^m h_1(z)$ for some $m \in \mathbb{Z}$, $h_1(z_0) \neq 0$.

$\therefore$ $f(z) = \frac{1}{(z-z_0)^m} f_1(z)$ where $f_1 = \frac{1}{h_1}$ and analytic in Δ and is analytic

So f has a pole at z_0

Thm (Caserati - Weierstrass)

If f is analytic in $\Delta^* \equiv \Delta^*(z_0, r)$ for some $r > 0$ and has an essential singularity at z_0, then $\overline{f(\Delta^*)} = \mathbb{C}$ (i.e. $\mathbb{C} \setminus f(\Delta^*)$ has no interior pts)

Proof Assume $w_0 \in (\mathbb{C} \setminus f(\Delta^*))^\circ$. $\exists\, \delta > 0$, $\forall z \in \Delta^*$, $|f(z) - w_0| > \delta$.

$\therefore$ $h(z) := \frac{1}{f(z) - w_0}$ is bounded in Δ^* (by $\frac{1}{\delta}$) and analytic in Δ^*.

So h has a removable singularity so can be extended to an analytic fn $\hat{h}$ of $\Delta(z_0, r)$. So $\frac{1}{\hat{h}}$ is either analytic or has a pole of finite order at 0. but this is $f(z) - w_0$. $\times$

Picard's Theorem. If f has an essential singularity at z_0 then in a punctured disk $\Delta^* = \Delta^*(z_0, r)$ (for $r > 0$), $|\mathbb{C} \setminus f(\Delta^*)| \leq 1$.

Defn a function f has 'no worse than a pole' at z_0 if it doesn't have an essential singularity at z_0.

Theorem if f and g have no worse than a pole at z_0 then

f', $f+g$, fg has no worse than a pole at z_0. The same is true for f/g if g is not identically zero.

<u>Defn</u> f is <u>meromorphic</u> in $U \overset{open}{\subseteq} \mathbb{C}$ if it has no essential singularities in U.

<u>eg</u> any rational function in $\frac{\mathbb{C}[x]}{\mathbb{C}[x]}$, meromorphic in $\mathbb{C}$.

<u>eg</u> $\tan z$ meromorphic in $\mathbb{C}$.