Lec 3/19

Monday, March 19, 2018 10:19

Recall at a singularity z_0

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Define Principal part of Laurent series

$$S(z) = \sum_{n=-\infty}^{-1} a_n (z - z_0)^n$$

If $S(z) = 0$, removable singularity: i.e. $\frac{\sin z}{z}$.

when $S(z)$ has only a finite number of terms,

z_0 a pole

the term a_{-1} is the residue of $f(z)$ at $z = z_0$.

Formula for residue of a pole of order m.

$$\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m S(z) \right) \Big|_{z = z_0} = a_{-1}$$

$$\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z) \right) \Big|_{z = z_0} = a_{-1}$$

Can determine type of pole by multiplying by $(z - z_0)^m$

and taking limit. (must get a finite nonzero limit).

residue is limit if $m=1$.

$$(z-1)^{-3} \cos\left(\frac{\pi}{2} z\right) = -\frac{1}{\xi^3} \sin\left(\frac{\pi}{2}\xi\right) \approx \frac{c\,\xi}{\xi^3}$$

expect a pole of order 2 at $\xi = 0$ $(z=1)$.

what's the residue?

1st method

$$\frac{d}{d\xi}\left(\xi^2\left(-\frac{1}{\xi^3} \sin\left(\frac{\pi}{2}\xi\right)\right)\right)\Bigg|_{\xi=0}$$

$$= \left(-\frac{\sin\left(\frac{\pi}{2}\xi\right)}{\xi}\right)' = \frac{-\frac{\pi}{2}\cos\left(\frac{\pi}{2}\xi\right)}{\xi} + \frac{1}{\xi^2}\sin\left(\frac{\pi}{2}\xi\right)$$

use L'Hop to take limit. (twice),

alternatively, compute taylor / Laurent series.

Residue $= 0$.

$z^2 e^{-1/z^2}$ at $z=0$

$$\sum^{2} (-1)^{\left(\frac{2-n}{2}\right)} z^n$$

$$\sum_{\substack{n=-\infty \\ (n \text{ even})}} \frac{\ }{\left(\frac{z-n}{2}\right)!}$$

$$S(z) = \sum_{n=-\infty}^{-1} 0$$

So $z=0$ is a essential singularity

residue is thus 0 ($n=-1$ coefficient).

$$\frac{1}{(\text{Arctan}\, z)^2} = \frac{1}{\left(z - \frac{z^3}{3} + \cdots\right)^2} = \frac{1}{z^2(1-O(z^2))^2}$$

Note: $\frac{1}{(1-w)^2} = 1 + 2w + 3w^2 + \cdots$

(derivative of g-series). for $|w|<1$.

here $w = O(z^2)$

So this is $\frac{1}{z^2}\left(1 + 2(O(z^2)) + 3(O(z^2))^2 + \cdots\right)$

$$= \frac{1}{z^2}\left(1 + O(z^2)\right)$$

$$= \frac{1}{z^2} + O(1)$$

so residue is 0, $S(z) = \frac{1}{z^2}$.

Compute the singular part of $\dfrac{\sin(z^3)}{(1-\cos(z))^3}$ at $z=0$.

$$= \frac{z^3 - \frac{z^9}{3!} + \frac{z^{15}}{5!} + \cdots}{\left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots\right)^3}$$

$$= \frac{z^3\left[1 - \frac{z^6}{3!} + \frac{z^{12}}{5!} + \cdots\right]}{\frac{z^6}{2^3}\left(1 - \frac{z^2}{12} + \frac{z^4}{360} + \cdots\right)^3}$$

$$= \frac{8}{z^3} \frac{(1 + O(z^6))}{(1-w)^3}$$

$$(1-w)^{-3} = 1 + 3w + O(w^2)$$

$$w = 1 - \frac{z^2}{12} + O(z^4)$$

$$\text{so } (1-w)^{-3} = 1 + 3 - \frac{z^2}{12} + O(z^8)$$

$$= 4 - \frac{z^2}{12} + O(z^8)$$

$$\text{So its } \frac{8}{z^3}\left[1 + O(z^6)\right]\left[4 - \frac{z^2}{12} + O(z^8)\right]$$

$$= \frac{8}{z^3}\left[4 - \frac{z^2}{12} + O(z^6)\right]$$

$$= \underbrace{\frac{32}{z^3} - \frac{2}{3}z^{-1}}_{S(z)} + O(z^3)$$

So residue is $-\frac{2}{3}$.