Lec 2/7

Wednesday, February 7, 2018 10:20

Recall thm: $\exists$ a branch of Log f (f analytic in D) iff

$\forall$ pwsc γ $\int_\gamma \frac{f'(z)}{f(z)}\,dz = 0$. $\quad g(z) = \mathrm{Log}(z_0) + \int_\alpha \frac{f'(z)}{f(z)}\,dz$

Any branch differs from g by $2\pi i k$.

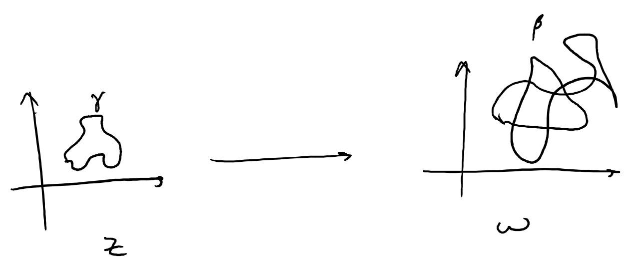
Suppose G is another branch, then

$$e^{G(z)} = f(z) = e^{g(z)} \Rightarrow e^{G(z)-g(z)} = 1$$

$$\Rightarrow G(z) - g(z) = 2\pi i k.$$

geometric Interpretation. if $\gamma = \gamma(t)$, $a \le t \le b$,

Define $\beta(t) = f(\gamma(t))$.



now $\int_\gamma \frac{f'(z)}{f(z)}\,dz = \int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)\,dt$

$$= \int_\beta \frac{dw}{w} = 2\pi i\, n(\beta, 0)$$

So condition is that $f(D)$ must not "circle around the origin" so that the winding # of β must be 0.

Corollary: for a rational function

$$f(z) = a(z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_r)^{m_r}$$

where $m_j \in \mathbb{Z} \setminus \{0\}$.
$1 \le j \le r$

then a branch of $\log f$ exists in a domain D iff $\forall$ pwsc γ, $m_1 n(\gamma, z_1) + m_2 n(\gamma, z_2) + \cdots + m_r n(\gamma, z_r) = 0$.

Proof: Use the last theorem:

$$\frac{f'(z)}{f(z)} = \frac{m_1}{z-z_1} + \frac{m_2}{z-z_2} + \cdots + \frac{m_r}{z-z_r}$$

so

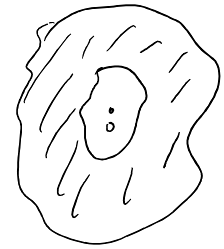
$$0 = \int_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i\left(m_1 n(\gamma, z_1) + \cdots + m_r n(\gamma, z_r)\right)$$

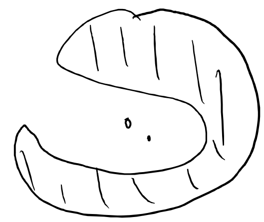
eg take $f(z) = z$. Last corollary becomes $n(\gamma, 0) = 0 \ \forall \gamma \subseteq D$.
So it must not be possible to circle the origin.

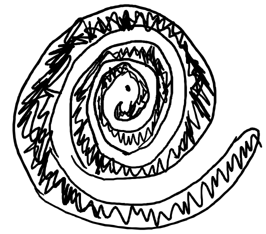
Does a branch of $\log z$ exist for D_1, D_2, D_3?

D:

No

D_1:  No

D_2:  Yes

D_3:  yes.

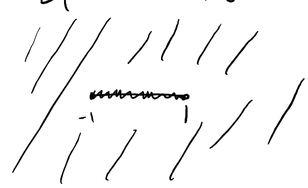
formula is $\log z = \operatorname{Log} z_0 + \int_\alpha \frac{dz}{z}$

where α from z_0 to z.

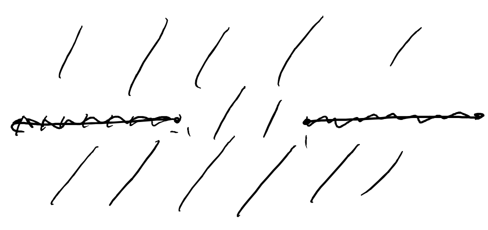
<u>eg</u> give examples of domain D for which a branch of $\log\left(\frac{z+1}{z-1}\right)$ exists.

from corollary, $n(\gamma, 1) = n(\gamma, -1)$ must hold.

one ex: $D_1 = \mathbb{C} \setminus [-1, 1]$.



another one



$D_2 = \mathbb{C} \setminus ((-\infty, 1] \cup [1, \infty))$.

formula (for D_1): $$\log\left(\frac{z+1}{z-1}\right) = \operatorname{Log}(3) + \int_\alpha \frac{dz'}{z'+1} - \int_\alpha \frac{dz'}{z'-1} + 2ik\pi$$

where α from 2 to z

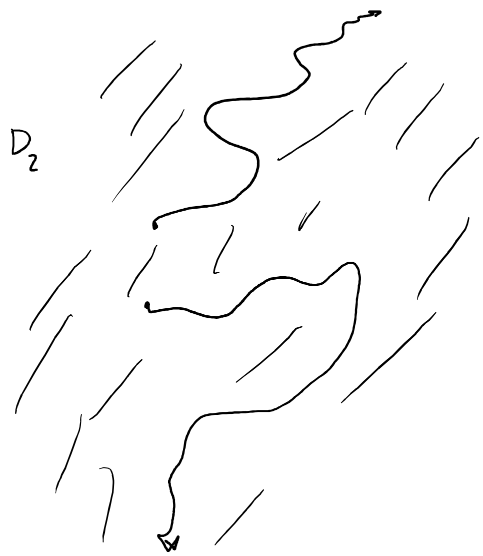
Determine domains D for which arctan exists

recall: $$\arctan z = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) = \frac{1}{2i}\log\left(-\left(\frac{z-i}{z+i}\right)\right)$$

Need $n(\gamma, i) = n(\gamma, -i)$

So domains are D_1

D_2



D_2 if you want arctan to be analytic @ origin.

Branch of $(f(z))^\lambda$ for $\lambda \in \mathbb{C}$.

Any domain D for a branch of $\operatorname{Log} f$ will also work as a domain for a branch of $(f(z))^\lambda$ since $(f(z))^\lambda = e^{\lambda \operatorname{Log} f(z)}$

However, one can have a domain D where $(f(z))^\lambda$ has a branch but $\log f$ does not.

eg $(z^2)^{1/2}$. one branch is z, other is $-z$.

both are analytic in $\mathbb{C}$, but $\log(z^2)$ is not.

Determine a suitable domain D for which $\sqrt{z^2-1}$ has a branch.

$z^2-1 = (z-1)(z+1)$ so we can have $n(\gamma, 1) + n(\gamma, -1) = 0$

$\forall \gamma \in D$. so $D_1 = \mathbb{C} \setminus ((-\infty, 1] \cup [1, \infty))$.

this is the domain where $\log(z^2-1)$ has a branch.

What about $D_2 = \mathbb{C} \setminus [-1, 1]$? there is

no branch of $\log(z^2-1)$ here.

but $\sqrt{z^2-1} = (z-1)\sqrt{\frac{z+1}{z-1}} = (z-1)\exp\left(\frac{1}{2}\log\left(\frac{z+1}{z-1}\right)\right)$

and this <u>is</u> good on D_2!