Lec 2/6

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Hadamard

Let $S = \{z : 0 < \operatorname{Re} z < 1\}$. Assume f is analytic in S and cont. on $\bar{S}$ (and that f is bdd on $\bar{S}$), and that $|f(iy)| \le m_0$, $|f(1+iy)| \le m_1$. then $\forall x \in (0,1)$,

$$|f(x+iy)| \le m_0^{1-x} m_1^{x}$$

Proof:

Case 1: $m_0 = m_1 = 1$.

Introduce $g(z) = \dfrac{f(z)}{1+tz}$ for some $t>0$, $\quad |1+tz| = |1+tx+ity|$

$$= \sqrt{(1+tx)^2 + t^2y^2} \ge ty$$

$(1+tx)^2 \ge 1$ since $x \ge 0$

So $g(z) \to 0$ as $|y| \to \infty$

Then $\forall$ fixed $t>0$ we know $\exists\, y_0 > 0$ s.t. $|g(z)| \le 1$ for $|\operatorname{Im} z| \ge y_0$

g is analytic in S & cts in its closure. Let $D =$

$|g| \le 1$ on all boundaries of D, and so

by maximum principle $|g| \le 1$ on D

and so $|g| \le 1$ on S. thus $|f| \le |1+tz|$.

Since t was arbitrary, $|f| \le 1$.

Note: we don't need f bounded, we

could take f "algebraically bounded" meaning

$f(z) \leq p(z)$ for some $p(z) \in \mathbb{C}[z]$.

we could replace tz by tz^{n+1} where n is degree of $P(z)$.

Case 2: $m_0 \neq 0 \neq m_1$ are arbitrary.

then first consider $\tilde{f}(z) = \frac{f(z)}{m_0^{1-z} m_1^z}$. This is analytic in S & cts in $\bar{S}$. And if $z = iy$ then

$|m_0^{1-z} m_1^z| = m_0^{1-x} m_1^x = m_0$. if $z = 1+iy$

then $|m_0^{1-z} m_1^z| = m_1$. So $|\tilde{f}(iy)| \leq 1$, $|\tilde{f}(1+iy)| \leq 1$.

So $\tilde{f}$ satisfies conditions in pt. 1. then

$|\tilde{f}(z)| \leq 1 \quad \forall z$ so $|f(z)| \leq |m_0^{1-z} m_1^z| = m_0^{1-x} m_1^x$.

iii) $m_1 = 0$ or $m_0 = 0$. We replace m_0 by $m_0 + \varepsilon$, and m_1 by $m_1 + \varepsilon$.

to use case 2 to get $|f(z)| \leq (m_0+\varepsilon)^{1-x}(m_1+\varepsilon)^x \to 0$ as $\varepsilon \to 0$.

Branches of Log of $f(z)$ analytic

Def g is a branch of $\operatorname{Log} f$ in a domain D if g is analytic in D, and $e^{g(z)} = f(z)$ $\forall z \in D$.

Remark Not a good idea to think of $\operatorname{Log} f$ as a composition $\operatorname{Log} \circ f$.

eg $f(z)=e^z$, $\log(e^z) = \overbrace{z + 2\pi i k}^{\text{no restriction needed}}$. However if you think of the composition you'll make unnecessary cuts.

Theorem. $\exists$ a branch of $\operatorname{Log} f$ in a domain D iff $\underbrace{\forall \text{ piecewise smooth closed } \gamma \subseteq D, \int_\gamma \frac{f'(z)}{f(z)}\,dz = 0.}_{(*)}$

Proof: Assume $(*)$ holds. Then $\exists$ a primitive of $\frac{f'(z)}{f(z)}$, call it $g(z)$ so $g'(z) = \frac{f'(z)}{f(z)}$. Define $g(z)$ to be

$$\operatorname{Log} \underbrace{f(z_0)}_{\substack{\text{note} \\ f \text{ is never } 0}} + \int_\alpha \frac{f'(\zeta)}{f(\zeta)}\,d\zeta \quad \text{where } \alpha \subseteq D \text{ goes from } z_0 \text{ to } z.$$

Now $\frac{d}{dz}\left(e^{-g(z)} f(z)\right) = 0$ and $e^{-g(z_0)} f(z_0) = 1$, $\square$

Assume g is a branch of $\operatorname{Log} f$.

then $\frac{d}{dz} g(z) = \frac{f'(z)}{f(z)}$ so g is a primitive for $\frac{f'(z)}{f(z)}$ so $(*)$ holds $\square$