Lec 2/5

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Thm if f analytic in $U \sim \{z_0\}$ & cts in U, it is analytic in U.

Thm (Liouville) a bounded entire function is constant.

Proof. Take any $z_0 \in \mathbb{C}$. $f'(z_0) = \frac{1}{2\pi i} \oint_{K(z_0,\rho)} \frac{f(z)}{(z-z_0)^2}\,dz$.

Let $|f(z)| \le M \ \forall z$. We have

$$|f'(z)| \le \frac{1}{2\pi} \oint_{K(z_0,\rho)} \frac{|f(z)|}{|z-z_0|^2}\,|dz| \le \frac{M}{2\pi\rho^2} \oint_{K(z,\rho)} |dz| = \frac{M}{\rho} \longrightarrow 0 \text{ as } \rho \to \infty.$$

so $f'(z_0) = 0$. since z_0 arbitrary, f constant $\square$

Fundamental Theorem of Algebra

Any polynomial $p(z) \in \mathbb{C}[z]$ has a root unless $p(z)$ constant

Proof: define $f(z) = \frac{1}{p(z)}$. If p has no roots, $f(z)$ is a bounded entire function so f is a constant. $\square$

(note $f(z) \to 0$ as $|z| \to \infty$, so it's bdd outside of a circle & cts inside (on a compact set) & so bounded).

Max Modulus thm.

If f is analytic in a domain $D \subset \mathbb{C}$ then if $|f|$ attains a max value M at an interior pt, then f is constant.

Proof: let $z_0 \in D$ be where $f(z)$ attains its max.
take $\Delta = \Delta(z_0, r)$ s.t. $\overline{\Delta} \subset D$. Then C.I. formula gives

$$f(z_0) = \frac{1}{2\pi i} \oint_{K(z_0,r)} \frac{f(z)\,dz}{z - z_0}.$$ So ~~new~~ where $M = f(z_0)$

$$M = |f(z_0)| \le \frac{1}{2\pi} \oint_0^{2\pi} |f(z_0 + re^{it})|\,|dt| \le \frac{1}{2\pi} \oint_0^{2\pi} M\,|dt| = M$$

So we must have equality throughout so

$$\oint_0^{2\pi} |f(z_0 + re^{it})| - M \quad dt = 0,$$ but that integrand is nonpositive & continuous, so it must be 0, $|f(z + re^{it})| = M$.

this is true $\forall r$, so f is constant on $\Delta(z_0, r)$.

Now this means $U = \{z \in D : |f(z)| = M\}$ is open, so is

$V = \{z \in D : |f(z)| < M\}$ since f is continuous & this is inv. image of an open set.

but D is connected so $V = \emptyset$. So $|f(z)| = M \ \forall z \in D$.

Now by C-R equations, f is constant. $\square$

Corollary If D is a bounded domain & $f : \overline{D} \to \mathbb{C}$ is cts & $f : D \to \mathbb{C}$ is analytic, then $|f|$ takes its cts & $f: D \to \mathbb{C}$ is analytic, then $|f|$ takes its max value on $\partial D = \overline{D} \setminus D$.

Pf if f constant, obvious. Now suppose f non-constant.
$|f|$ cts on compact set so takes a max, but f analytic on D so does not take a max there. boom.

If $|f|$ attains max at $z_0 \in D$, f constant & so max is also attained on boundary. D.

Schwarz Lemma:

Assume f is analytic in $\Delta(0,1)$ and cts in $\overline{\Delta}$. and $f(0)=0$ and $|f(z)| \leq 1 \ \forall z \in \Delta(0,1)$.
Then i) $|f'(0)| \leq 1$. ii) $\left|\frac{f(z)}{z}\right| \leq 1$. iii) unless $f(z) = e^{i\phi}z$, $\left|\frac{f(z)}{z}\right| < 1$.

Pf $g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$ is analytic on $\Delta(0,1)$ ($\frac{f(z)}{z}$ analytic everywhere except 0, g cts).

then $|g(z)|$ attains max on boundary of $\overline{\Delta}$, which is $\leq \frac{1}{1} = 1$

So i & ii follow. if $f(z) = e^{i\phi}z$, g is constant so we don't have strict inequality. o.w. we do.