Lec 2/28

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<u>Recall</u> for $\{r_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$, $\limsup_{n\to\infty} r_n = \lim_{n\to\infty} \sup_{j \geq n} r_n$

If r_n is bdd, $\limsup r_n = \max$ acc. pts of r_n

<u>Property</u>: if $r = \limsup r_n$, only finitely many $r_n > r + \varepsilon$, and infinitely many $r_n > r - \varepsilon$

Define for a series of the form $S = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ $\rho(S) = \left(\limsup_{n\to\infty} |a_n|^{1/n}\right)^{-1}$

$\left(\frac{1}{\infty} = 0, \ \frac{1}{0} = \infty\right)$

<u>Thm</u>: (i) S diverges for $|z - z_0| > \rho$.

(ii) S converges absolutely & normally for $|z - z_0| < \rho$ to an analytic fn $f(z)$ and $\frac{f^{(k)}(z_0)}{k!} = a_k$.

<u>Proof</u>: (i) if $r = |z - z_0| > \rho$, notice that $\frac{1}{\rho^n} > \frac{1}{r^n}$ for $n \in \mathbb{N}$. Since $\rho^{-1} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$, so $\exists$ infinitely many a_n for which $|a_n| > \frac{1}{r^n}$, so infinitely many n have $|a_n (z - z_0)^n| > 1$ so series diverges

(ii) if $|z - z_0| \leq r < \rho$ Now except for some finite number of terms, $|a_n|^{1/n} < \frac{1}{\rho}$ so $|z - z_0|^n |a_n| \leq \frac{1}{\rho^n} |z - z_0|^n < \left(\frac{r}{\rho}\right)^n$

since $\frac{r}{\rho} < 1$, M-test applies & so $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is uniformly & absolutely convergent for $|z-z_0| \le r$, so $f(z)$ is analytic.
the fact that $\frac{f^{(k)}(z_0)}{k!} = a_k$ follows from BASIC shit.

eg examine convergence & divergence of

a) $\sum \frac{z^n}{n}$ b) $\sum \frac{n!}{(2n)!} (z+i)^n$

c) $\sum 2^n z^{n^2}$ d) $\sum n! \, z^n$

Comment: ratio test and root test still works.

a) $\lim_{n\to\infty} \left|\frac{1}{n}\right|^{\frac{1}{n}} = 1$ so converges if $|z| < 1$.

b) Apply ratio test:
$$\frac{\frac{(n+1)!}{(2n+2)!} (z+i)^{n+1}}{\frac{n!}{(2n)!} (z+i)^n} = \frac{(2n+2)(2n+1)}{n+1} |z+i| \to 0$$

so converges for $|z| < \infty$.

c) $\sum 2^n z^{n^2} = \sum a_k z^k$

but apply root test. $\left(2^n |z|^{n^2}\right)^{1/n} = 2|z|^n \to \infty$ if $|z|>1$, $\to 0$ if $|z|<1$

so converges for $|z| < 1$.

d) never converges except at $z=0$.

Thm Suppose f is analytic in some open $U \subset \mathbb{C}$ and $z_0 \in U$ and $\Delta(z_0, r) \subset U$

then for $z \in \Delta(z_0, r)$, $f(z) = \sum a_n (z-z_0)^n = \sum \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$.

Pf take $z \in \Delta(z_0, r)$. Choose s so that $|z-z_0| < s < r$.

then Cauchy integral formula $\Rightarrow f(z) = \frac{1}{2\pi i} \int_{K(z_0, r)} \frac{f(\zeta)}{(\zeta - z_0)} d\zeta$

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0}\left(1 - \frac{z-z_0}{\zeta - z_0}\right)^{-1}$$

$$\frac{|z-z_0|}{|\zeta - z_0|} < \frac{s}{r} < 1.$$

geom. series.

$$= \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{\zeta - z_0}\right)^n$$

$$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n \frac{1}{2\pi i} \int_{K(z_0, r)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

$$= \sum \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$$

Remark: if $f(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$ convergent then

by thm before last, $b_n = \frac{f^{(n)}(z_0)}{n!}$.

eg $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$.

eg

$$g(z) = \begin{cases} \dfrac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

$$g(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}$$

eg taylor series of $\frac{1}{1-z^2}$ at $z = i$

$$\frac{1}{1-z^2} = \frac{1}{2} \frac{1}{1-z} + \frac{1}{2} \frac{1}{1+z}$$

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i}\left(1 - \frac{z-i}{1-i}\right)^{-1}$$

$$||$$

so $\frac{1}{2(1-i)} \sum \frac{(z-i)^n}{(1-i)^n}$