# Lec 2/26

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<u>Def</u> convergence of series $\sum_{n=1}^{\infty} z_n \iff$ convergence of $S_n = \sum_{k=1}^{n} z_k$.

<u>Defn</u> A sequence $\sum_{n=-\infty}^{\infty} z_n$ is said to converge if $S_{m,n} = \sum_{k=-m}^{n} z_n \xrightarrow{\text{to } S}$ converges

in the sense that given $\varepsilon > 0$, $\exists N$ s.t. if $m, n > N$, $|S_{m,n} - S| < \varepsilon$

<u>Thm</u> $\sum_{k=-\infty}^{\infty} z_k$ converges iff both $\sum_{k=0}^{\infty} z_k$ and $\sum_{k=-\infty}^{-1} z_k$ both converge.

Pf: $S_{m,n} = \sum_{k=0}^{n} z_k + \sum_{k=1}^{m} z_{-k}$

<u>Defn</u> $\sum_{k=0}^{\infty} z_k$ converges absolutely if $\sum_{k=0}^{\infty} |z_k|$ converges.

<u>Thm</u> Absolute convergence $\Rightarrow$ convergence.

<u>pf</u> triangular inequality on tail. $S_m - S_n$.

<u>eg</u> consider $\sum_{n=1}^{\infty} \frac{i^n}{n} = \sum_{n=1}^{\infty} \frac{i^{2n-1}}{2n-1} + \sum_{n=1}^{\infty} \frac{i^{2n}}{2n}$

$$= \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{2n} - i \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}}_{}$$

Converge, but $\sum |\frac{i^n}{n}| = \sum \frac{1}{n}$ div.

<u>Thm</u> if $z_n = x_n + iy_n$, $\sum z_n$ conv. iff $\sum x_n$, $\sum y_n$ conv.

<u>Thm</u> if $\sum z_n$ abs. conv., $\sum z_{\sigma(n)}$ abs conv.

All tests for absolute convergence work.

eg consider $\sum_{n=-\infty}^{\infty} 2^{-|n|} z^n$. determine the set of $z$ for which this converges

$$\sum_{n=0}^{\infty} \frac{z^n}{2^n} \qquad \sum_{n=1}^{\infty} \frac{1}{(2z)^n}$$

abs conv. for $|z| < 2$. abs conv. for $|z| > \frac{1}{2}$.

So $\sum_{n=-\infty}^{\infty} 2^{-|n|} z^n$ conv. absolutely in annulus $\{z \in \mathbb{C} : \frac{1}{2} < |z| < 2\}$.

Consider: $\sum_{n=1}^{\infty} f_n(z)$. Can define (i) point-wise convergence
(ii) Uniform convergence
(iii) Normal Convergence

in terms of convergence of $S_n(z) = \sum_{k=1}^{n} f_k(z)$.

Theorem a necessary condition for pointwise convergence of a series $\sum_{k=1}^{\infty} f_k(z)$ is that $\lim_{k\to\infty} f_k(z) = 0 \ \forall z$.

Theorem (Weierstrass M-test)

Suppose $f_n$ is defined in $A \subseteq \mathbb{C}$ and for every $z \in A$, $|f_n(z)| \leq M_n$.

Then if $\sum_{n=1}^{\infty} M_n < \infty$ then $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly (1) and absolutely (2) in $A$.

Proof: Consider $\{S_n(z) = \sum_{k=1}^{n} f_k(z)\}_{n=1}^{\infty}$. It is enough to show that $(S_n)$ is uniformly Cauchy in $A$ for (1) eg $\sum_{n=1}^{\infty} \frac{1}{n^z} = \zeta(z)$. Show it converges unif. in $A_\sigma = \{z : \operatorname{Re} z \geq \sigma > 1\}$.

$$\left|\frac{1}{n^z}\right| = |n^{-z}| = n^{-x} \leq \frac{1}{n^\sigma}$$ and $\sum \frac{1}{n^\sigma}$ converges since $\sigma > 1$.

<u>Thm</u> Suppose $f_n$ is continuous in an open set $U \subseteq \mathbb{C}$ and $\sum_{n=1}^{\infty} f_n(z)$ converges normally in $U$. Then $f(z) = \sum_{n=1}^{\infty} f_n(z)$ is continuous in $U$ and $\int_\gamma f(z)\,dz = \sum_{n=1}^{\infty} \int_\gamma f_n(z)\,dz$ $\forall$ pwsc $\gamma$ in $U$.