Lec 2/16

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Harmonic functions in 2-D

If D is a domain and $u(x,y) \in C^2(D)$ (i.e. has cts 2nd. der. wrt x,y) for $z = x+iy \in D$.

And u satisfies $u_{xx} + u_{yy} = 0$, u is called harmonic in D.

Theorem If $f(z)$ is analytic in D then both $u(x,y)$ and $v(x,y)$ (where $f(x+iy) = u(x,y) + iv(x,y)$) are harmonic.

Proof Cauchy-riemann: $u_x = v_y$, $u_y = -v_x$.

So $u_{xx} = v_{yx}$, $u_{yy} = -v_{xy}$ since f' is analytic so $u, v \in C^2$

but $v_{yx} = v_{xy}$ so $u_{xx} = -u_{yy}$ so $u_{xx} + u_{yy} = 0$.

Similarly. $u_{xy} = v_{yy}$ and $u_{yx} = -v_{xx}$ so $v_{xx} + v_{yy} = 0$. $\square$

Defn if $u + iv = f$ where $z = x+iy$ and f is analytic in D,

then v is defined to be the harmonic conjugate of u

eg $e^x \sin y + c$ is harmonic conjugate of $e^x \cos y + d$.

Since $e^{x+iy} + d + ic$ is analytic.

Notice if v is h.c. of u then $-u$ is h.c. of v.

Since $-if$ is analytic

Theorem: Let D be a domain. Every harmonic function in D

has a harmonic conjugate in D iff D is simply connected.

Proof Assume first that D is simply connected. Consider function $g = u_x - i u_y$. Notice the CR conditions are satisfied by g: $u_{xx} = -u_{yy}$, and $u_{xy} = u_{yx}$. So g is analytic in D.

Define f to be a primitive of g, which exists since D simply connected. Then $f' = u_x - i u_y$.

Let $f = U + iV$, and then $f' = U_x - iU_y$. So $u = U + c$, and so $V + d$ is a harmonic conjugate to u. $\blacksquare$

Now assume each u has a harmonic conjugate in D.

Take $u = \ln|z - z_0|$ for some $z_0 \in \mathbb{C} \setminus D$. $u = \operatorname{Re} \operatorname{Log}(z - z_0)$

[So $\operatorname{Log}(z - z_0)$ is analytic in $\mathbb{C} \setminus D$.] Let $f = u + iv$ be analytic.

Consider $h(z) = (z - z_0) e^{-f(z)}$. $h'(z) = e^{-f(z)} \left(1 + (z - z_0)(-f'(z))\right)$.

but $f'(z) = u_x - i u_y = \frac{1}{z - z_0}$ (exercise) so $h'(z) = 0$, so

h is constant in D, so $(z - z_0) e^{-f(z)} = c$ so $e^{f(z)} = c(z - z_0)$

so f is a branch of $\log(z - z_0) + C$ which is analytic in D.

So D is simply connected.