Lec 2/13

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<u>Lemma</u> if $z \in$ any connected component of $\mathbb{C} \setminus |\sigma|$, then $n(z,\sigma) = \sum_{i=1}^{\ell} n(z,\gamma_i)$ is a constant. Pf: each one stays constant.

<u>Defn</u>: a cycle σ contained in an open set U is homologous to 0 in U if $\forall z \in \mathbb{C} \setminus U$, $n(\sigma, z) = 0$

<u>Defn</u> two cycles σ_1, σ_2 are homologous to each other in an open set U if $\sigma_1 - \sigma_2$ is homologous to 0 in U.

<u>Global Cauchy Theorem</u>

Assume $U \subset \mathbb{C}$ is open. Then $\int_\sigma f(z)\,dz = 0$ for all f analytic in U iff σ is homologous to 0 in U.

<u>Proof</u>: Assume $\int_\sigma f(z)\,dz$ is true $\forall f$ analytic in U. Thus if $\zeta \in \mathbb{C} \setminus U$, $f(z) = \frac{1}{2\pi i}\frac{1}{z-\zeta}$ is analytic in U. So $n(\sigma,\zeta) = \int_\sigma f(z)\,dz = 0$, and so σ is homologous to 0.

Now assume σ is homologous to 0. Define $V = \{z \in \mathbb{C} \setminus |\sigma| : n(z,\sigma) = 0\}$

Now if $z \in V$ then it belongs to some connected component of $\mathbb{C} \setminus |\sigma|$ where n is a constant $= 0$, so V is open.

So $K = \mathbb{C} \setminus V$ is closed (and bounded since $V \supset$ unbdd comp)

and V doesn't contain $|\sigma|$ so $|\sigma| \subset K$.

Now we know $n(z,\sigma) = 0$ for $z \in \mathbb{C} \setminus U$, so $\mathbb{C} \setminus U \subset V$

so $K = \mathbb{C} \setminus V \subset U$. So $\exists \delta$ s.t. $\forall z \in K$, $\Delta(z,\delta) \subset U$.

Choose a grid $x = n\frac{\delta}{2}$ $y = m\frac{\delta}{2}$, $n, m \in \mathbb{Z}$.

Since K is compact, it is covered by finitely many (closed) grid squares w/ side length $\frac{\delta}{2}$. Only enumerate these finitely many, $Q_1, Q_2, \ldots, Q_r$ where $K \cap Q_i \neq \emptyset$.

Now if we draw a circle Δ_i w/ same center as Q_i and radius $\frac{\delta}{2}$, Δ_i will contain Q_i, and $\Delta_i \subset U$.

NOW we can use Cauchy integral formula.

Suppose $z \in K$. Then $z \in Q_m$ for some m.

Case a: $z \in Q_m^\circ$. Then $\frac{1}{2\pi i} \int_{\partial Q_m} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z)$.

Also for $j \neq m$, $\frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(\zeta)}{\zeta - z} d\zeta = 0$

Adding up $\frac{1}{2\pi i} \sum_{j=1}^{r} \int_{\partial Q_j} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z)$.

Thus all lines which intersect w/ K are cancelled, and this sum is $\frac{1}{2\pi i} \sum_{j=1}^{M} \int_{\lambda_j} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z)$ where λ_j are the boundary edges of the boxes.

In fact, we can lump in case b ($z \in \partial Q_m$) since end result "doesn't care", by continuity.

Now $\int_\sigma f(z)\,dz = \int_\sigma \frac{1}{2\pi i} \sum_{j=1}^{M} \int_{\lambda_j} \frac{f(\zeta)}{\zeta - z} d\zeta\, dz = \sum_{j=1}^{M} \int_{\lambda_j} d\zeta f(\zeta) \underbrace{\frac{1}{2\pi i} \int_\sigma \frac{dz}{\zeta - z}}_{= -n(\sigma, \zeta) = 0}$

since ζ in unbdd comp of $\mathbb{C} \setminus |\sigma|$.