Lec 1/9

Tuesday, January 9, 2018 10:22

$z = |z| (\cos\theta + i\sin\theta)$

$w = |w| (\cos\varphi + i\sin\varphi)$

$wz = |w||z| (\cos(\theta+\varphi) + i\sin(\theta+\varphi))$

$|wz| = |w||z|$

$\arg(wz) = \arg w + \arg z$ (Note $\text{Arg}(wz) \neq \text{Arg}\, w + \text{Arg}\, z$

__Lemma__ $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ for $n \in \mathbb{N}$.

__Proof__: $n=2$: $(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta) = \cos(2\theta) + i\sin(2\theta)$ (from last time).

Rest by induction.

(De Moivre's theorem)

__Theorem__ For $n \in \mathbb{N}$, the n^{th} root of $z = |z|(\cos\theta + i\sin\theta)$ is given by

$$|z|^{1/n}\left(\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right) \quad \text{for } k = 0, 1, \dots, n-1.$$

__Proof__: Suppose $w^n = z$. $w = |w|(\cos\varphi + i\sin\varphi)$. Thus using lemma,

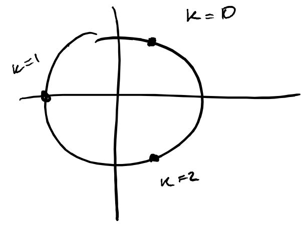
$$w^n = |w|^n (\cos(n\varphi) + i\sin(n\varphi)) = |z| (\cos\theta + i\sin\theta)$$

$$\Rightarrow |w|^n = |z| \Rightarrow |w| = |z|^{1/n} \text{ and } n\varphi = \theta + 2k\pi \text{ for } k \in \mathbb{Z}.$$

__eg__ Determine all roots of $z^3 + 8 = 0$. $z^3 = -8 = 8(\cos\pi + i\sin\pi)$

so $z = 2\left(\cos\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)\right)$ for $k = 0, 1, 2$.

geometrically. The roots are uniformly spaced out on a circle of radius 2



Determine $\sqrt{2+\sqrt{1+i}}$. $\sqrt{1+i} = 2^{1/4}(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}), -2^{1/4}(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8})$

add 1, convert to polar again, find root.

<u>Defn</u>. If $y \in \mathbb{R}$, $e^{iy} \equiv \sum_{n=0}^{\infty} \frac{(iy)^n}{n!}$

<u>Lemma</u>. $e^{iy} = \cos y + i \sin y$.

Let $S_N = \sum_{n=0}^{N} \frac{(iy)^n}{n!}$. $\left|e^{iy} - S_N\right| \leq \sum_{n=N+1}^{\infty} \frac{|iy|^n}{n!} = \sum_{n=N+1}^{\infty} \frac{|y|^n}{n!}$

we know (for real t) that $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$ converges absolutely

So the tail of seq. tends to 0. Let $t = |y|$. so S_N converges

as $N \to \infty$, so e^{iy} is well-defined.

Since it's abs. conv we can separate:

$$e^{iy} = \sum_{n\geq 0}^{\infty} \frac{(iy)^n}{n!} = \sum_{\substack{n \text{ even} \\ = \\ 2m}} \frac{(-1)^m y^{2m}}{(2m)!} + i \sum_{\substack{n \text{ odd} \\ = \\ 2m+1}} \frac{(-1)^{2m+1} y^{2m+1}}{(2m)!} = \cos y + i \sin y.$$

<u>Corollary</u>. if $y, t \in \mathbb{R}$, $e^{iy}e^{it} = e^{i(y+t)}$

<u>Pf</u> use product of sin & cos formula.

<u>Defn</u> if $z = x+iy$, let $e^z = e^x e^{iy}$

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Lemma. if $z, w \in \mathbb{C}$, $e^z e^w = e^{z+w}$

Pf. let $z = x+iy$, $w = s+it$.

$$e^z e^w = e^x e^{iy} e^s e^{it} = e^{x+s} e^{i(y+t)} = e^{w+z}.$$

Corollary: $\frac{1}{e^z} = e^{-z}$. because $e^{-z}e^z = e^0 = 1$.

Lemma If $e^z = e^w$ then $z-w = 2i\pi k$ for $k \in \mathbb{Z}$.

Pf $e^z e^{-w} = 1 \Rightarrow e^{z-w} = 1$. write $z-w = x+iy$.

$$e^{z-w} = e^x(\cos y + i\sin y) \Rightarrow e^x = 1 \Rightarrow x=1$$

$$\cos y + i\sin y = 1 \Rightarrow y = 2k\pi.$$

Def. $\operatorname{Log} z \equiv \ln|z| + i\operatorname{Arg} z$ (principal branch of Logarithm)

lemma $e^{\operatorname{Log} z} = z$. pf. $e^{\ln|z| + i\operatorname{Arg} z} = e^{\ln|z|} e^{i\operatorname{Arg} z} = |z| e^{i\operatorname{Arg} z} = z$.

lemma. if $e^w = z$ then $w = \operatorname{Log} z + 2ik\pi$ for $k \in \mathbb{Z}$.

or $w = \ln|z| + i\arg z$

Proof $e^w = z = e^{\operatorname{Log} z} \Rightarrow w - \operatorname{Log} z = 2k\pi i$.

Defn. $\log z = \ln|z| + i\arg z$ (each k is a diff "branch" of log).

Lemma. $\log(z_1 z_2) = \log z_1 + \log z_2$ (Note: $\mathrm{Log}\, z_1 z_2 \neq \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2$).

Proof: let $w_1 = \log z_1$, $w_2 = \log z_2$. $z_1 = e^{w_1}$, $z_2 = e^{w_2}$, $z_1 z_2 = e^{w_1+w_2}$

so $\log(z_1 z_2) = w_1 + w_2$.

Defn. z^λ is well-defined for $\lambda \in \mathbb{C}$, $z \neq 0$.

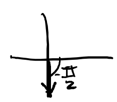
$z^\lambda \equiv e^{\lambda \log z}$. Principal branch of z^λ is $e^{\lambda \mathrm{Log}\, z}$.

ex: Calculate principal value of i^i.

$= e^{i\,\mathrm{Log}\, i}$. $i = e^{i\pi/2}$, so $e^{i\,\mathrm{Log}\, i} = e^{i(\ln 1 + i\pi/2)} = e^{-\pi/2}$.

$(zw)^\lambda = z^\lambda w^\lambda$ w/ branch unspecified but not true for principal branch.

eg Principal branch: $(-i)^{1/2} \neq (-1)^{1/2} \cdot i^{1/2}$.



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$e^{-i\pi/4} = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$.

$(-1)^{1/2} = i$

$i^{1/2} = e^{i\frac{\pi}{4}}$ $\Rightarrow$ RHS $= \frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}} \neq$ LHS (they are opposites).