Lec 1/8

Sunday, January 7, 2018 23:24

Complex nos. as ordered pairs of two real nos (a,b)
equipped with two operations: $+$ and $*$.

(1) $(a,b) + (c,d) = (a+c,\ b+d)$

(2) $(a,b) * (c,d) = (ac-bd,\ ad+bc)$

$\mathbb{C}$ is a field. $0 = (0,0)$, $1 = (1,0)$

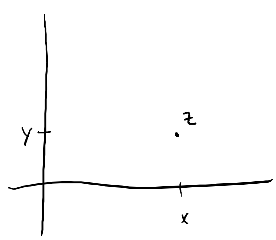
$-(a,b) = (-a,-b)$

$(a,b)^{-1} = \left(\frac{a}{a^2+b^2},\ -\frac{b}{a^2+b^2}\right)$ for $(a,b) \neq (0,0)$.

$\{(x,0) : x \in \mathbb{R}\} \subset \mathbb{C}$ is a subfield isomorphic to $\mathbb{R}$.

<u>Geometric interpretation</u>

$z = (x,y)$ $\quad x = \text{Re } z \quad y = \text{Im } z$



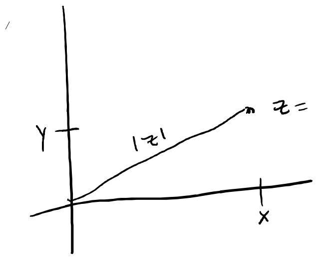
Note: $(0,1)^2 = (-1,0) = (0,-1)^2$

$i \equiv (0,1)$

$-i = (0,-1)$

$(x,y) = (x,0) + (0,y) = x + iy$

$$(a+ib)(c+id) = ac + iad + ibc + i^2 bd$$
$$= (ac-bd) + i(ad+bc)$$



$|z| = \sqrt{x^2+y^2}$

↑ absolute or modulus of z.

<u>Defn</u> $z = x+iy$, $\bar{z} = x-iy$ (complex conjugate).

Properties: $\operatorname{Re} z = \dfrac{z+\bar{z}}{2}$ $\quad \operatorname{Im} z = \dfrac{z-\bar{z}}{2i}$

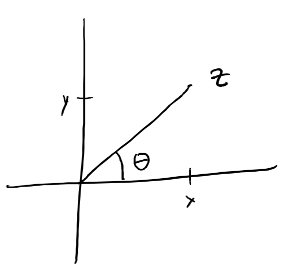
$$|z|^2 = z\bar{z} \qquad \bar{z} = z^*$$

$$\overline{z \pm w} = \bar{z} \pm \bar{w}$$

$$\bar{\bar{z}} = z \qquad \overline{zw} = \bar{z}\,\bar{w}$$

$$\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}} \quad \text{so} \quad \overline{w^{-1}} = \bar{w}^{-1}$$

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{z\bar{z}}$$



argument of z.

θ not unique: could pick $\theta + 2\pi k$.

for any $z \in \mathbb{C}$, we can find θ so that $\overset{x+iy}{\overset{\parallel}{z}} = \underbrace{|z|(\cos\theta + i\sin\theta)}$

choose θ:

$$\cos\theta = \frac{x}{|z|}, \quad \sin\theta = \frac{y}{|z|}.$$

polar form of z.

θ not unique unless restricted to 2π period.

$\theta \equiv \arg z$

if we restrict $\theta \in (-\pi, \pi]$ then it is the principal agument.

written as $\operatorname{Arg} z$
$\uparrow$
capital A.

$$\arg \bar{z} = -\arg z$$

Ex: if $z = 2+3i$

determine $|z|$, $\operatorname{Arg} z$

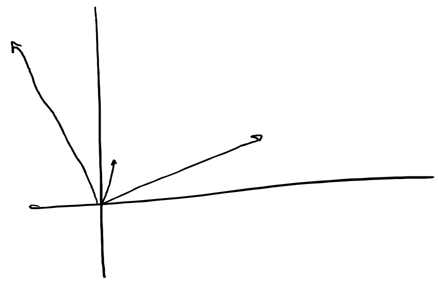
$|z| \downarrow \sqrt{2^2+3^2}$, $\operatorname{Arg} z \downarrow \arctan\left(\frac{3}{2}\right)$

if $z = -2-3i$, $\operatorname{Arg} z = \arctan\left(\frac{3}{2}\right) - \pi$

If $z = |z| (\cos\theta + i\sin\theta)$

$w = |w| (\cos\varphi + i\sin\varphi)$

then
$$zw = |z||w| (\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi)$$
$$= |z||w| \left((\cos\theta\cos\varphi - \sin\theta\sin\varphi) + i(\sin\theta\cos\varphi + \cos\theta\sin\varphi) \right)$$
$$= |z||w| \left(\cos(\theta+\varphi) + i\sin(\theta+\varphi)\right)$$



$$\arg\left(\frac{1}{z}\right) = \arg\left(\frac{\bar z}{|z|^2}\right) = \arg \bar z = -\arg z$$

$$\frac{z}{w} = \frac{|z|}{|w|}\left(\cos(\theta-\varphi) + i\sin(\theta-\varphi)\right).$$

btw, $|zw| = |z||w|$.

triangular inequality:

$$|z+w| \le |z|+|w|$$

$$|z+w|^2 = (z+w)(\overline{z+w}) = (z+w)(\bar z+\bar w) = z\bar z + w\bar w + z\bar w + w\bar z$$
$$= |z|^2+|w|^2 + 2\operatorname{Re} z\bar w$$

and $\operatorname{Re} z\bar w \le |z\bar w| = |z||w|$

so this is $\le |z|^2+|w|^2 + 2|z||w| = (|z|+|w|)^2$.

and $|z \pm w| \geq \big||z|-|w|\big|$.