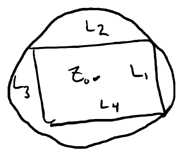
Lec 1/31

Wednesday, January 31, 2018 10:18

<u>Recall</u> if f is cont. in some $\triangle$ and analytic in $\Delta \setminus \{z_0\}$.
then $\forall$ pwsc $\gamma \subset \Delta$, $\int_\gamma f(z)\,dz = 0$. (Local Cauchy Thm)

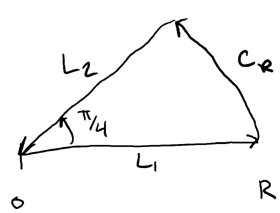
<u>eg</u>:



$$\int_{\partial R} \frac{1}{z-z_0}\,dz = \int_{\text{circle}} \frac{1}{z-z_0}\,dz = 2\pi i .$$

$$I \quad \int_0^\infty \cos(t^2)\,dt = \operatorname{Re} \underbrace{\int_0^\infty e^{it^2}\,dt}_{I_1}$$

Consider



$$\int_{L_1+C_R+L_2} e^{iz^2}\,dz = 0$$

$$\int_{C_R} e^{iz^2}\,dz \quad . \quad \text{on } C_R,\ z = Re^{it},\quad 0 \le t \le \pi/4 .$$

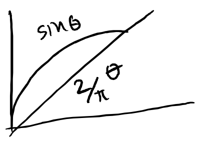
$$\| $$

$$\int_0^{\pi/4} e^{iR^2(\cos 2t + i\sin 2t)}\, iRe^{it}\,dt = \int_0^{\pi/4} e^{-R^2\sin 2t + iR^2\cos 2t}\, iRe^{it}\,dt$$

$$\left|\int_{C_R} e^{iz^2}\,dz\right| \le \int_0^{\pi/4} \left| e^{-R^2\sin(2t) + iR^2\cos 2t}\, iRe^{it}\right| dt = \int_0^{\pi/4} e^{-R^2\sin 2t}\, R\,dt$$

$$// \quad \pi/2 \qquad \pi/2$$

$$\theta = 2t \rightarrow 2R\int_0^{\pi/2} e^{-R^2 \sin\theta}\, \partial\theta \;\le\; 2R \int_0^{\pi/2} e^{-R^2 \frac{2}{\pi}\theta}\, \partial\theta$$



$$= -\frac{\pi\, e^{-R^2 \frac{2}{\pi}\theta}}{R}\Bigg]_0^{\pi/2}$$

$$= \frac{\pi}{R}\left[1 - e^{-R^2}\right] \longrightarrow 0 \text{ as } R \to \infty.$$

Now $\int_{L_2} e^{iz^2}\, \partial z$. L_2 is param. by $\gamma(r) = r e^{i\pi/4}$, $\partial z = e^{i\pi/4}$ as r from R to 0.

$z^2 = r^2 i \Rightarrow iz^2 = -r^2$.

$$\int_{L_2} e^{iz^2}\,\partial z = e^{i\pi/4}\int_R^0 e^{-r^2}\,\partial r = -e^{i\pi/4}\int_0^R e^{-r^2}\,\partial r \longrightarrow -e^{i\pi/4}\frac{\sqrt{\pi}}{2}$$

so $$\int_{L_1} e^{iz^2}\,\partial z \longrightarrow e^{i\pi/4}\frac{\sqrt{\pi}}{2} \text{ as } R \to \infty.$$

So $$\int_0^\infty \cos(t^2)\,\partial t = \operatorname{Re} e^{i\pi/4}\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{2\sqrt{2}} = \sqrt{\frac{\pi}{8}}$$

What about $\int_0^\infty \sin(t^2)\,\partial t$? it is also $\frac{\sqrt{\pi}}{2\sqrt{2}}$.

$$\int_{-\infty}^{\infty} \cos t^2\, dt = 2\int_0^\infty \cos t^2\, dt = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

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$$\int_{-\infty}^{\infty} \sin t^2 \, dt = 2 \int_0^{\infty} \sin t^2 \, dt = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Prove that if f is analytic in a domain Δ containing a simple closed path γ traversed ccw. then for z_0 inside γ,

then $\displaystyle\int_\gamma \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$

Note:

$$\int_\gamma \frac{f(z)}{z-z_0}\,dz = \underbrace{\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz}_{\text{must show this is } 0} + f(z_0)\underbrace{\int_\gamma \frac{dz}{z-z_0}}_{= 2\pi i.}$$

define $g(z) = \begin{cases} \dfrac{f(z)-f(z_0)}{z-z_0} & z \neq z_0. \\ f'(z_0) & z = z_0. \end{cases}$ this is continuous

and analytic everywhere other than at z_0. So the integral is 0 and the result follows.

Winding #s:

Defn: for a closed piecewise smooth γ, define for $z \in \mathbb{C} \setminus |\gamma|$

$$n(z,\gamma) = \frac{1}{2\pi i}\int \frac{d\xi}{\xi - z} = \frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t)-z}\,dt$$

Lemma: $n(z,\gamma)$ is necessarily an integer.

Proof: define $g(t) = \displaystyle\int_a^t \frac{\gamma'(s)}{\gamma(s)-z}\,ds$.

except for $t = t_0, t_1, \dots, t_k$ where γ' might not exist,

we know $g'(t) = \dfrac{\gamma'(t)}{\gamma(t)-Z}$

two $n(z,\gamma) = \dfrac{1}{2\pi i}\, g(b)$

consider $\frac{d}{dz}\Big(\overbrace{(\gamma(t)-z)\ e^{-g(t)}}^{h(t)}\Big)$ for $t \in$ some (t_{k-1}, t_k)

$$= \gamma'(t)e^{-g(t)} + (\gamma(t)-z)(-g'(t))\,e^{-g(t)} = 0$$

so h is constant on all intervals & cts so h is constant in $[a,b]$ so $h(t) = h(a)$ so $h(b) = h(a)$

but $\gamma(a) = \gamma(b)$ too so $e^{-g(b)} = e^{\overbrace{-g(a)}^{0}}$ so $g(b) = 2\pi k i$.

So $n(z,\gamma) = K \in \mathbb{Z}$

allow $\arg z \in (0, \infty)$ to be continuous (not necessarily 1-1)

then winding # is just difference in argument.