Lec 1/29

Monday, January 29, 2018 10:22

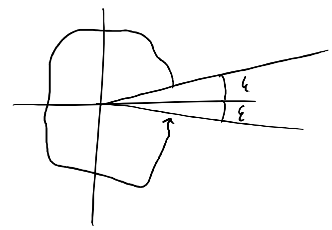
Ex Show that $\forall$ pws simple closed curve γ containing 0,

$$\int_\gamma \frac{1}{z}\,dz = 2\pi i$$

Consider

$$\int_{\gamma_\varepsilon} \frac{1}{z}\,dz$$

where γ_ε :



$\log z = \ln|z| + i\arg z$

$0 < \arg z < 2\pi$

$$= \log z = \ln|z| + i\arg z \Big|_{\text{endpts}}$$

We take open U containing γ_ε where $\log z$ is analytic

by continuity,

$$\int_\gamma \frac{1}{z}\,dz = \lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} \frac{1}{z}\,dz$$

$$\|$$

$$\lim_{\varepsilon \to 0} \ln|z| + i\arg z \Big|_{\text{endpts}}$$

$$= 0 + 2\pi i - 0i = 2\pi i.$$

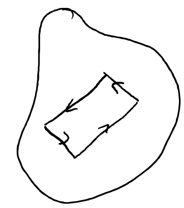
Can replace piecewise smooth by rectifiable:

$\forall$ partition $a = t_0 < t_1 < \cdots < t_n = b$

$$\sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})| \text{ is bounded}$$

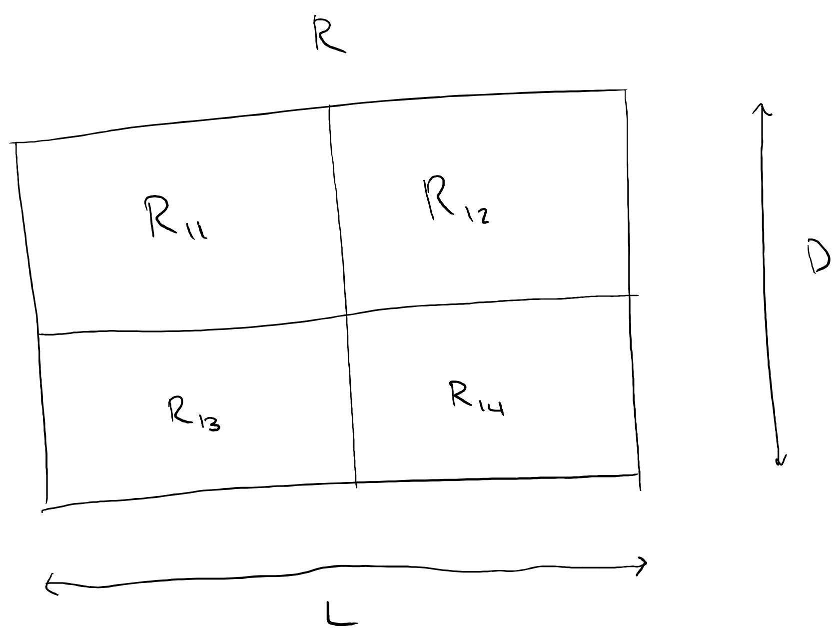
Cauchy's Theorem

Lemma 1 Suppose f is analytic in some open $U \subseteq \mathbb{C}$. Then $\forall$ closed rectangle $R \subseteq U$, $\int_{\partial R} f(z)\,dz = 0$.



Proof:

Subdivide R into 4 Rectangles



Notice:

$$\int_{\partial R} = \int_{\partial R_{11}} + \int_{\partial R_{12}} + \int_{\partial R_{13}} + \int_{\partial R_{14}}$$

since overlapping paths cancel.

Claim: $\left|\text{one of the 4 contributions on the right}\right| \geq \frac{1}{4}\left|\int_{\partial R}\right|$.

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call the contributing 'tangle R_1.

We write

$$\left|\int_{\partial R_1}\right| \geq \frac{1}{4}\left|\int_{\partial R}\right|$$

Repeat argument for R on R_1 to obtain R_2 etc.

and we have

$$\left|\int_{\partial R_j} f(z)dz\right| \geq \frac{1}{4^j}\left|\int_{\partial R} f(z)dz\right|$$

the dimensions of R_j : $\underset{\text{length}}{\frac{L}{2^j}} \times \underset{\text{width}}{\frac{D}{2^j}}$

And by construction $R \supset R_1 \supset R_2 \supset R_3 \supset \cdots$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \cdots$

Non empty compact sets.

from cantor's thm, $\exists\, z_0 \in \bigcap_{j=1}^{\infty} R_j$

Now since f is analytic, given $\varepsilon > 0$ $\exists$ $c \in \mathbb{C}$, $\delta > 0$ s.t. for $z \in \Delta(z_0, \delta)$, $f(z) = f(z_0) + c(z - z_0) + E(z)$

with $|E(z)| \leq |z - z_0|\varepsilon$

Now find some $R_j \subset \Delta(z_0, \delta)$. We have

$$\left|\int_{\partial R} f(z)\,dz\right| \leq 4^j \left|\int_{\partial R_j} f(z)\,dz\right| = 4^j \left|\int_{\partial R_j} f(z_0) + c(z-z_0) + E(z)\,dz\right|$$

$$= 4^j \left| \underset{\nwarrow}{0} + \underset{\nearrow}{0} + \int_{\partial R_j} E(z)\,dz \right|$$

have primitives we know

$$\leq 4^j \int_{\partial R_j} |E(z)|\,|dz|$$

$$\leq 4^j \int_{\partial R_j} \varepsilon |z-z_0|\,|dz|$$

$$= LD\varepsilon \frac{4^j}{4^j} = LD\varepsilon$$

which shows the lemma. □

<u>Lemma 2</u> If f is continuous on an open set U and analytic in $U \setminus \{z_0\}$ for some $z_0 \in U$, then $\int_{\partial R} f(z)\,dz = 0$ $\forall$ rectangles R.

<u>Proof</u> Again subdivide R into 4^j rectangles, $R_j \ni z_0$. And using some bounds we get $\int_{\partial R_j} = 0$. (read later).

<u>Lemma 3</u> If $D \subset \mathbb{C}$ is an open disk and f is ctg in D and $\forall\ R \subset D$ $\int f(z)\,dz = 0$ then f has a primitive in D

$\forall\ R \subset \Delta \quad \int_{\partial R} f(z)\,dz = 0$ then f has a primitive in Δ

and $\int_{\gamma} f(z)\,dz = 0 \quad \forall$ p.w.s. closed $\gamma \subset \Delta$.