Integration on a complex curve:

$\int f(z)\,dz$ on the directed curve.

Defn $\gamma : [a,b] \to \mathbb{C}$ is a path if it's continuous.

eg line $\gamma(t) = (1-t)z_0 + tz_1$, $0 \le t \le 1$.

eg half-circle $\gamma(t) = 1 + 2e^{it}$

Defn: smooth path if $\frac{d\gamma}{dt} = \dot{\gamma}$ exists and is cts. for $t \in (a,b)$

if $\gamma = x + it$, $\dot{\gamma} = \dot{x} + i\dot{y}$.

Notation $[z_0, z_1]$ is straight line curve $\overrightarrow{z_0 z_1}$. (invariant of path representation)

Piecewise Smooth: if $\exists$ a partition of $[a,b]$: $a = t_0 < t_1 < t_2 < \cdots < t_n = b$

s.t. γ is diff able on (t_k, t_{k+1}) $\forall k$.

eg: $\gamma(t) = \begin{cases} 1 + 2e^{it} & 0 \le t \le \pi \\ (1-(t-\pi))(-1) + (t-\pi)(3) & \pi \le t \le \pi + 1 \end{cases}$

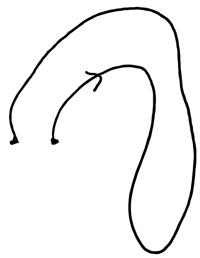
Sim.

Defn: $-\gamma$ is defined as the path $-\gamma(t) = \gamma(b+a-t)$

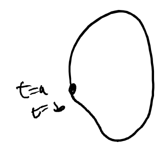
(reversing the direction of the path).

Defn γ is said to be simple if $\gamma(t)=\gamma(s) \Rightarrow t=s$ (except possibly when $\{t,s\}=\{a,b\}$).

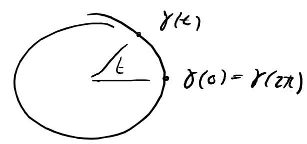
eg



eg



eg: $\gamma(t)=e^{it}$, $\gamma:[0,2\pi]\longrightarrow \mathbb{C}$



Defn γ is closed (as a path) if $\gamma(a)=\gamma(b)$.

if $\gamma(t)=t$ then the path is $[a,b]$.

Defn: if $g:[a,b]\longrightarrow \mathbb{C}$, $\underset{\text{is continuous}}{g(t)=u(t)+iv(t)}$, we define

$$\int_a^b g(t)\,dt \equiv \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

If g is piecewise cts, we have the same defn.

or rather $\displaystyle\int_a^b g(t)\,dt = \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} g(t)\,dt$

Thm: if $\exists\ G\in C^1[a,b]$ s.t. $g(t)=G'(t)$

then $\displaystyle\int_a^b g(t)\,dt = G(b)-G(a)$

$$\int_a g(t)\,dt = G(b) - G(a)$$

eg: $$\int_0^{\pi} e^{it}\,dt = -i e^{it}\Big|_0^{\pi} = i + i = 2i$$

Defn if f is cts on path γ, i.e. $f \circ \gamma$ is continuous on $[a,b]$ and γ is smooth on $[a,b]$. Then:

$$\int_\gamma f(z)\,dz \equiv \int_a^b f(\gamma(t))\,\dot{\gamma}(t)\,dt$$

If γ is piecewise smooth then

$$\int_\gamma f(z)\,dz \equiv \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(\gamma(t))\,\dot{\gamma}(t)\,dt$$

(don't see how this defn is necessary, previous case covers this).

Note if $f(z) = u(x,y) + i v(x,y)$ (where $z = x + iy$)

$$\int_\gamma f(z)\,dz = \int_C (u + iv)(dx + i\,dy)$$

$$= \int_C (u\,dx - v\,dy) + i \int_C (u\,dy + v\,dx)$$

Arc length integral:

Defn: if f is cts on a piecewise smooth path then

$$\int_\gamma f(z)\ |dz| \equiv \int_a^b f(\gamma(t))\ |\dot\gamma(t)|\, dt$$

Note: if $f(z)=1$ then $\int_\gamma |dz| = \int_a^b |\dot\gamma(t)|\, dt$ gives the arc length of γ on $[a,b]$.

changing parameterization doesn't change integral

Lemma

$$\left| \int_\gamma f(z)\, dz \right| \leq \int_\gamma |f(z)|\ |dz|$$

Proof $z = \gamma(t)$. LHS is $\left| \int_a^b f(\gamma(t))\ \dot\gamma(t)\, dt \right| \leq \int_a^b |f(\gamma(t))|\, |\dot\gamma(t)|\, dt$

$$= \int_\gamma |f(z)|\ |dz|$$