Lec 1/22

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Thm If $f'(z) = 0$ for $z \in D$ then $f(z) = c$

Pf Recall that for $r > 0$ small enough, for any z_0, we showed

$f(z) = f(z_0)$ for $z \in \Delta(z_0, r) \Longrightarrow U = \{z \in D : f(z) = f(z_0)\}$ is an open set.

$V = \bigcup_{\substack{z_1: \\ f(z_1) \neq f(z_0)}} \{z \in D : f(z_1) = f(z)\} = \{z \in D : f(z) \neq f(z_0)\}$ is open,

now $V \cap U = \phi$ and $C = U \cup V$ so $V = \phi$.

Thm If f is analytic in D and either i) $\operatorname{Re} f = c$

ii) $\operatorname{Im} f = c$

iii) $|f| = c$

Then f is constant in D

Proof i) $f(x+iy) = u(x,y) + i\,v(x,y)$

C.R. $v_y = u_x \Rightarrow 0 = v_y$

$v_x = -u_y \Rightarrow 0 = -v_x$

So v is const

so f is const.

ii) $u_x = v_y = 0$, $u_y = -v_x = 0$, u is const, f is const.

iii) $|f(z)|^2 = u^2(x,y) + v^2(x,y) = c^2$

then $2uu_x + 2vv_x = 0$ (∂ wrt x)

$\Rightarrow uu_x - vu_y = 0$

$$u u_y + v v_y = 0$$

$$u u_y + v u_x = 0$$

Solve system $\Rightarrow$ $(u^2+v^2)\, u_x = 0$

$$(u^2+v^2)\, u_y = 0$$

If $c=0$ nothing to show. if not, $u_x = u_y = 0$ so u is const.

Similarly $v_x = v_y = 0$ so v is const.

Def z_0 is called a critical point of f if f is analytic at z_0 and $f'(z_0)=0$.

eg Determine critical points of $f(z) = e^{-1/z} + z$ analytic in $\mathbb{C} \setminus \{0\}$

$$f'(z) = e^{-\frac{1}{z}+z}\left(\frac{1}{z^2}+1\right) = 0 \Rightarrow z = \pm 1.$$

Defn If $f : U \longrightarrow \mathbb{C}$ is analytic in $U \subseteq \mathbb{C}$ then $g : D \longrightarrow \mathbb{C}$ $(D \subseteq f(U))$ is said to be a branch of f^{-1} in Domain D if g is continuous and $(f \circ g)(w) = w$.

eg $f(z) = e^z$. $g(z) = \operatorname{Log} z + 2ki\pi$ is an inverse $\forall k$.

Thm If $f : U \to D$ is analytic and $g : D \to U$ is a branch of f^{-1} in D then g is analytic at any pt $w_0 = f(z_0)$ if $f'(z_0) \neq 0$.

Pf Note g is univalent i.e. $g(z_1)=g(z_2) \Rightarrow z_1=z_2$ since we know $(f \circ g)(w_2) = (f\circ g)(w_1) \Rightarrow w_1 = w_2$.

if $f(z)=w$, $f(z_0)=w_0$

$$\frac{g(w)-g(w_0)}{w-w_0} = \frac{z-z_0}{f(z)-f(z_0)} \longrightarrow \frac{1}{f'(z)} = \frac{1}{f'(g(w_0))}$$

Trig functions

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz}-e^{iz}}{2i}$$

$$\cosh z = \frac{e^{z}+e^{-z}}{2}$$

$$\sinh z = \frac{e^{z}-e^{-z}}{2}$$

Similarly, tan, sec, cot, csc, tanh, sech, cos

$$\cosh(iz) = \cos(z)$$

Det set of all z s.t. $\cos z = 2$. $\frac{e^{iz}+e^{-iz}}{2} = 2$

$$\xi = e^{iz} \quad \xi^{-1} = e^{-iz} \Rightarrow \frac{\xi + \frac{1}{\xi}}{2} = 2$$

$$\Rightarrow \quad \xi = -2 \pm \sqrt{3} .$$

$$-iz = \operatorname{Log}(-2+\sqrt{3})$$

$$= \ln(2-\sqrt{3}) + i\pi + 2ki\pi$$

eg Determine mapping properties of $f(A)$ where $f(z) = \sin z$ and $A = \{ z : -\frac{\pi}{2} < \operatorname{Re} z < \pi/2 , \operatorname{Im} z > 0 \}$

f univalent

inverse $\quad g(w) = -i \operatorname{Log}((1-w^2)^{1/2} + iw)$

Principal branch

Trig identities hold still (can use defn to prove this).