Lec 1/19

Friday, January 19, 2018 10:59

Cauchy-Riemann Condition

Show $\operatorname{Log} z$ is analytic in $U = \mathbb{C} \setminus \mathbb{R}^-$.

$$\operatorname{Log} z = \ln|z| + i\operatorname{Arg}(z)$$

$$= \underbrace{\tfrac{1}{2}\ln(x^2+y^2)}_{u(x,y)} + i \underbrace{\begin{cases} \arcsin\left(\frac{y}{|z|}\right) & \text{for } x > 0 \\ \pi - \arcsin\left(\frac{y}{|z|}\right) & \text{for } x \le 0,\ y > 0 \\ -\pi - \arcsin\left(\frac{y}{|z|}\right) & \text{for } x \le 0,\ y < 0. \end{cases}}_{v(x,y)}$$

Can check:

$u_x = v_y$, $\quad u_y = -v_x \quad$ for each case.

Also continuity of partials may be checked in U.

$\Longrightarrow$ $\operatorname{Log} z$ is analytic in U.

Know: $f'(z) = u_x + i v_x$

$$z = e^{\operatorname{Log} z}$$

$$1 = z' = e^{\operatorname{Log} z}\left(\frac{d}{dz}\operatorname{Log} z\right)$$

$$\frac{1}{z} = \frac{1}{e^{\operatorname{Log} z}} = \frac{d}{dz}\operatorname{Log} z$$

Side: ex: Show $\frac{d}{dz} e^z = e^z$

$$e^z = e^x\cos y + i e^x \sin y$$

check C.R. & cont.

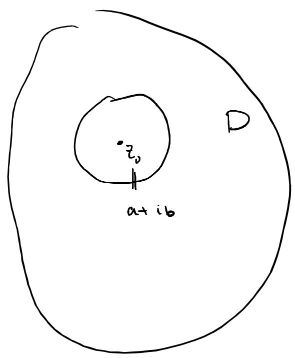
$$(e^z)' = u_x + i v_x = e^x \cos y + i e^x \sin y$$

eg Determine $\frac{d}{dz} z^k \quad \forall k.$ $\qquad z^k = e^{k \operatorname{Log} z}.$

$$\frac{d}{dz}(z^k) = e^{k \operatorname{Log} z}\,\frac{k}{z} = \frac{k z^k}{z} = k z^{k-1}. \qquad \text{for } z \in \mathbb{C} \setminus \mathbb{R}^-$$

<u>Theorem</u>. if D is a plane domain in $\mathbb{C}$ and $f'(z) = 0$ for $z \in D$, then $f(z) = c$, a constant.

<u>Pf</u>



$f(z) = u(x,y) + i\,v(x,y)$

$f(z) - f(z_0) = f'(c)\,(z - z_0) = 0.$

for some c on the polygonal path between z, z_0

So $f(z) = f(z_0)$ a constant.