Lec 1/18

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Ch 3: Derivative of $f: A \to \mathbb{C}$.

if z_0 is an interior pt of A, then $f'(z_0)$ exists when $\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$.

When it exists we call it $f'(z_0)$.

Lemma: $f'(z_0)$ exists iff $f(z) = f(z_0) + c(z - z_0) + E(z)$ (*) for some $c \in \mathbb{C}$

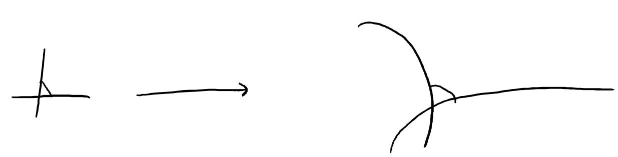
and some E with $\lim\limits_{z \to z_0} \dfrac{|E(z)|}{|z - z_0|} = 0$.

Proof: Assume (*). Then $\dfrac{f(z) - f(z_0)}{z - z_0} = c + \dfrac{E(z)}{z - z_0} \longrightarrow c$ $\blacksquare$

Assume $f'(z)$ exists. Then define $E(z) = f(z) - f(z_0) - f'(z_0)(z - z_0)$

then $c = f'(z_0)$. $\dfrac{E(z)}{z - z_0} = \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \longrightarrow 0$. $\blacksquare$

So diffable functions are locally linear. as long as $f'(z_0) = 0$.
(conformal)
thus angles under f are preserved.



Properties of derivative: If $f: A \to \mathbb{C}$, $g: A \to \mathbb{C}$ are diffable, $z_0 \in A^{\text{int}}$, then:

(i) $(cf)'(z_0) = c f'(z_0)$ (ii) $(f+g)'(z_0) = f'(z_0) + g'(z_0)$

(iii) $(fg)'(z_0) = f'(z_0) g(z_0) + f(z_0) g'(z_0)$

(iv) if $g(z_0) \neq 0$, $\left(f/g\right)'(z_0) = \dfrac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}$

eg: $f(z) = z^n$, $n \in \mathbb{Z}^+$. find $f'(z)$ and its domain set.

$$\lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0} = \lim_{z \to z_0}(z^{n-1} + z^{n-2}z_0 + \cdots + z_0^{n-1}) = nz^{n-1}. \quad \text{domain set: } \mathbb{C}.$$

$\frac{f}{g}(z) = z^{-n}$, $n \in \mathbb{Z}^+$.

$= \frac{1}{z^n}$

$$\frac{0 - nz^{n-1}}{z^{2n}} = -nz^{-n-1} \quad \text{for } z \neq 0. \text{ domain } \mathbb{C} \setminus \{0\}.$$

Cor if f' exists at $z_0 \in \text{int}(A)$.

Proof: $f(z) = f(z_0) + f'(z_0)(z - z_0) + E(z)$

take the limit as $z \to z_0$. it is $f(z_0)$.

Remark: Requirement that f has a derivative is far stronger than for real-valued function.

eg $f(z) = |z|^2 = x^2 + y^2$ (diffable in real sense)

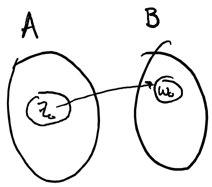
define $z - z_0 = h$. examine $\displaystyle\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$

let $z_0 = x_0 + iy_0$

If $h \in \mathbb{R}$, this is $\lim_{h \to 0} \dfrac{(x_0+h)^2 + y_0^2 - x_0^2 - y_0^2}{h} = 2x_0$

if $h \in i\mathbb{R}$ ($h = i\hat{h}$), this is $\lim_{\hat{h} \to 0} \dfrac{x_0^2 + (y_0+\hat{h})^2 - x_0^2 - y_0^2}{i\hat{h}} = -2iy_0$

Chain Rule **Thm** Let $z_0 \in \operatorname{int}(A) \subset \mathbb{C}$, $w_0 = f(z_0) \in \operatorname{int}(B) \subset \mathbb{C}$.



And assume that $\Delta(w_0, \hat{\delta}) \supset f(\Delta(z_0, \delta))$ (this is implied by continuity of f at z_0 since $f'(z_0)$ exists). assume $g'(w_0)$ exists as well.

Then $(g \circ f)'(z_0) = g'(f(z_0))f'(z_0)$

Proof: $\dfrac{(g \circ f)(z) - (g \circ f)(z_0)}{z - z_0}$ Define $w = f(z)$, $w_0 = f(z_0)$

$\quad\quad \|$

Want: $\dfrac{g(w) - g(w_0)}{w - w_0} \cdot \dfrac{w - w_0}{z - z_0}$

define: $G(w) = \begin{cases} \dfrac{g(w) - g(w_0)}{w - w_0} & w \neq w_0 \\ g'(w_0) & w = w_0 \end{cases}$ This is cts.

thus we get $G(w) \cdot \dfrac{w - w_0}{z - z_0} = G(w) \dfrac{f(z) - f(z_0)}{z - z_0} \longrightarrow g'(f(z_0))f'(z_0)$

$\dfrac{d}{dz}\left(\dfrac{z^n + 1}{z^n - 1}\right)$ can be computed