Lec 1/17

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__BWP:__ Bounded $\{z_n\}_{n=1}^{\infty}$ has at least 1 accumulation point, and there is exactly 1 accumulation point iff $z_n \to z_0$ converges.

__Cauchy Sequence:__ $\{z_n\}$ is a cauchy sequence if, given $\varepsilon > 0$, there is $N_\varepsilon \in \mathbb{N}$ so that $|z_n - z_m| < \varepsilon$ if $n, m > N_\varepsilon$

__Theorem:__ $\{z_n\}$ is cauchy iff it converges in $\mathbb{C}$.

__Defn:__ $\{z_n\}_{n=1}^{\infty}$ is contractive if $\exists\, \lambda \in (0,1)$ s.t. $|z_{n+2} - z_{n+1}| \leq \lambda |z_{n+1} - z_n| \quad \forall n \geq 1$.

__Lemma:__ Contractive sequences are Cauchy.

__Proof:__ by induction, $|z_{n+2} - z_{n+1}| \leq \lambda^n |z_2 - z_1|$

Now let $m > n \geq 1$ wlog.

$$\text{Then} \quad |z_m - z_n| \leq |z_m - z_{m-1}| + |z_{m-1} - z_{m-2}| + \cdots + |z_{n+1} - z_n|$$
$$\leq \left(\lambda^{m-2} + \lambda^{m-1} + \cdots + \lambda^{n-1}\right) |z_2 - z_1|$$
$$\leq \frac{\lambda^{n-1}}{(1-\lambda)} |z_2 - z_1|.$$

Thus choose N s.t. $\dfrac{\lambda^{N-1}}{1-\lambda} < \dfrac{\varepsilon}{|z_2 - z_1|}$.

__Compact sets__

__Defn:__ $K \subset \mathbb{C}$ is compact if every sequence $\{z_n\}$ in K has a subsequence $\{z_{n_k}\}$ so that $z_{n_k} \to z_0 \in K$.

Thm K is compact iff K is bounded & closed.

Pf: Assume K compact. Suppose K were unbounded.

take $\{z_n\} \in K^{\mathbb{N}}$ s.t. $|z_n| > n$. No subsequence exists which converges, $\cancel{\infty}$.

Suppose $z_0 \in \overline{K}$. Then $\exists \{z_n\} \in K^{\mathbb{N}}$ s.t. $z_n \to z_0$ as $n \to \infty$.

This sequence converges so any subsequence $z_{n_k} \to z_0$.

Thus $z_0 \in K$ by defn. of compactness, so $\overline{K} \subset K$.

Assume K closed & bounded. Let $\{z_n\} \in K^{\mathbb{N}}$. Since K bounded, $\{z_n\}$ has a subsequence which converges $z_{n_k} \to z_0$.

Since K is closed, $z_0 \in K$. Thus K is compact.

lemma Suppose $U \subset \mathbb{C}$ is open and $K \subset U$ is compact.

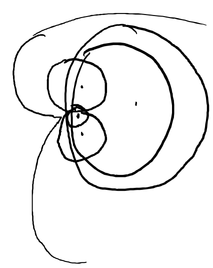
then $\exists r > 0$ s.t. $\Delta(z, r) \subset U$ $\forall z \in K$.

Pf. Suppose this is not the case. Then $\forall n \in \mathbb{N}$ $\exists z_n \in K$ and $w_n \in \mathbb{C} \setminus U$ s.t. $|z_n - w_n| < \frac{1}{n}$. Now $z_{n_j} \to z_0 \in K$ for some $n_j \nearrow \infty$.

And $|w_{n_k} - z_0| \leq |w_{n_k} - z_{n_k}| + |z_{n_k} - z_0| \leq \frac{1}{n_k} + |z_{n_k} - z_0| \longrightarrow z_0$

So $w_{n_k} \to z_0$ as well. But $\mathbb{C} \setminus U$ is closed so $z_0 \in \mathbb{C} \setminus U$.

A contradiction since $(\mathbb{C} \setminus U) \cap K = \emptyset$ since $K \subset U$.



Proof in class don't work.

Thm Cantor's Theorem:

Suppose $K_1 \supset K_2 \supset K_3 \supset \cdots$ are compact & non-empty.

then $\bigcap_{n=1}^{\infty} K_n$ is non-empty.

Proof choose $z_n \in K_n$. Notice $\{z_n\}_{n=1}^{\infty} \in K_1^{\mathbb{N}}$, so $\exists$ subseqn $z_{n_k} \to z_0 \in K_1$.
$z_0 \in \bigcap_{n=1}^{\infty} K_n$

Thm $f(K)$ is compact if K is compact.

pf Let $\bigcup_{\tau \in T} V_\tau$ be an open cover of $f(K)$. Then $\bigcup_{\tau \in T} f^{-1}(V_\tau)$ is an open cover of K. So there is a finite subcover $\bigcup_{n=1}^{s} f^{-1}(V_{\tau_n})$ of K. Then $\bigcup_{n=1}^{s} V_{\tau_n} \supset f(K)$ since if $z \in f(K)$ there is $w \in K$ s.t. $f(w) = z$, and $w \in$ some $f^{-1}(V_{\tau_n})$ so $z \in V_{\tau_n}$.

Cor: if $f : A \longrightarrow \mathbb{R}$ and $A \subset \mathbb{C}$ is compact, f attains a max & min value on A.

Ex: show that if $f : \mathbb{C} \longrightarrow \mathbb{C}$ is cts and $\lim_{|z| \to \infty} f(z) = 0$ then $|f(z)|$ has a max value on $\mathbb{C}$.