Lec 1/12

Friday, January 12, 2018 10:17

note Log is defined for $\mathbb{C}\setminus\{0\}$.

includes the origin

ex Show that $\operatorname{Log} z$ is a continuous function of z in $\mathbb{C}\setminus\{\mathbb{R}^-\} = D$

$$\operatorname{Log} z = \underbrace{\ln|z|}_{\text{cts}} + i\,\underbrace{\operatorname{Arg} z}_{\text{need to show this is cts here.}}$$

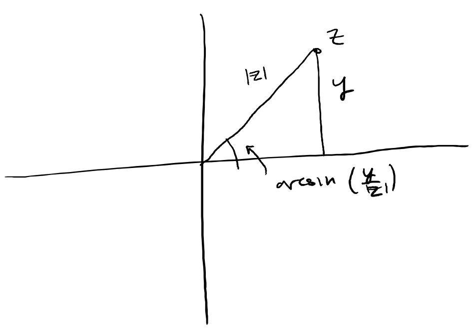
Define. $\alpha(\underset{x+iy}{z}) = \operatorname{Arcsin}\left(\frac{y}{|z|}\right)$. Note $\operatorname{Arcsin}(t)$ is a cts fn of $t \in [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$y = \operatorname{Im} z$ is cts in z in D. $|z|$ is cts in z in D. $0 \notin D$ so

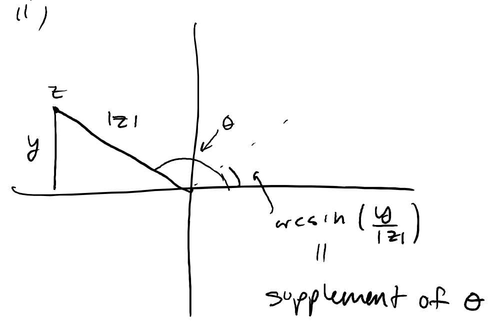
$\frac{y}{|z|}$ is cts in z in D. So $\alpha(z)$ is cts in D.

claim:
$$\theta(z) = \begin{cases} \alpha(z) & \text{when } x \geq 0 \\ \pi - \alpha(z) & \text{when } x < 0,\ y \geq 0 \\ -\pi - \alpha(z) & \text{when } x < 0,\ y < 0 \end{cases}$$

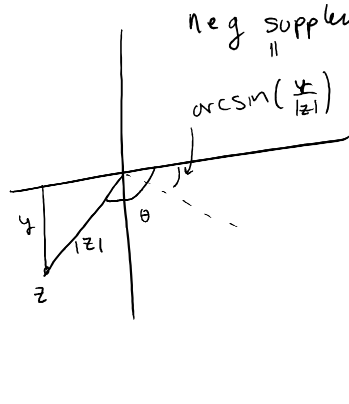
i)



ii)



iii)



We only need to check continuity on imaginary axis $i\mathbb{R}\setminus\{0\}$

Take $\operatorname{Im} z_0 > 0$. Then $\alpha(z) = \arcsin\left(\frac{y_0}{|z|}\right) = \frac{\pi}{2}$.

if $z \in \{z : \operatorname{Re} z > 0\}$, $|\theta(z) - \theta(z_0)| = |\alpha(z) - \alpha(z_0)|$

if $\operatorname{Re} z < 0$, $\operatorname{Im} z > 0$, $|\theta(z) - \theta(z_0)| = \left|\pi - \alpha(z) - \frac{\pi}{2}\right| = |\alpha(z) - \alpha(z_0)|$.

Also true for $\operatorname{Im} z < 0$. thus since α is cts, $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t.

$$|z - z_0| < \delta \Rightarrow |\theta(z) - \theta(z_0)| = |\alpha(z) - \alpha(z_0)| < \varepsilon.$$

Remark Note that $\frac{1}{z}$ is not discontinuous at $z = 0$. it is undefined at $z = 0$.

Thm if $U \subset \mathbb{C}$ is open then $f : U \longrightarrow \mathbb{C}$ is cts iff $\forall$ open $V \subset \mathbb{C}$, $f^{-1}(V)$ is also open.

Defn z_0 is a limit point of $A \subset \mathbb{C}$ if $\Delta^*(z_0, r) \cap A \neq \emptyset$ $\forall r > 0$.

Defn of Limit $\lim_{z \to z_0} f(z)$ where $f : A \to \mathbb{C}$. If $z_0 \in A$ then defn is obvious.

if z_0 is a limit point of A. Then $\lim_{z \to z_0} f(z) = w_0$ means that

$\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $f(\Delta^*(z_0, \delta) \cap A) \subset \Delta(w_0, \varepsilon)$.

Thm $f : A \longrightarrow \mathbb{C}$, $g : B \longrightarrow \mathbb{C}$ and z_0 is a limit point of $A \cap B$ where $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} g(z) = \hat{w}_0$, then

1) $\lim_{z \to z_0} (f \pm g)(z) = w_0 \pm \hat{w}_0$

2) $\lim_{z \to z_0} (cf)(z) = c w_0$ $\quad \forall c \in \mathbb{C}$

3) $\lim_{z \to z_0} (fg)(z) = w_0 \hat{w}_0$

4) $\lim_{z \to z_0} \frac{f}{g}(z) = \frac{w_0}{\hat{w}_0}$ if $\hat{w}_0 \neq 0$.

4) $\lim_{z \to z_0} \frac{f}{g}(z) = \frac{w_0}{\hat{w}_0}$ if $\hat{w}_0 \neq 0$.

Ex: Determine $\lim_{z \to 0} \dfrac{e^z}{1 + z \operatorname{Log} z}$

this is $\dfrac{\lim_{z \to 0} e^z}{\lim_{z \to 0} (1 + z \operatorname{Log} z)}$ provided both exist, bottom $\neq 0$.

$$= \frac{1}{1 + \lim_{z \to 0} z \operatorname{Log} z}$$

Consider $z \operatorname{Log} z$. Candidate $w_0 = 0$.

take $\varepsilon > 0$. then choose $\delta =$

$$\begin{aligned} |z \operatorname{Log} z - w_0| &= |z| \, |\ln|z| + i \operatorname{Arg} z| \\ &\leq |z| \sqrt{(\ln|z|)^2 + \pi^2} \\ &= r \sqrt{\ln^2 r + \pi^2} \end{aligned}$$

we know from real var thy that $\exists \, \delta_0$ s.t. if

$r < \delta_0$ then $r\sqrt{\ln^2 r + \pi^2} < \varepsilon$, so take $\delta = \delta_0$.

<u>Remark</u>: $g(z) = \dfrac{f(z) - f(z_0)}{z - z_0}$. Domain of g does not include z_0

if $z \in \operatorname{Dom}(f)$, still meaningful to ask if

$\lim g(z)$ exists.

$z \to z_0$

Remark: $e^{-1/x^2} \longrightarrow 0$ as $x \to 0$. However, $\lim\limits_{\substack{z \to 0 \\ \in \mathbb{C}}} e^{-1/z^2}$ DNE

(for real x)

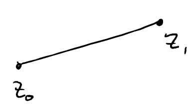
Defn Disconnected set $A \subset \mathbb{C}$. If $A = B \cup C$ where $\overline{B} \cap C = \emptyset = B \cap \overline{C}$.

Alt. if $\exists$ open $U, V \in \mathbb{C}$ s.t. $U \cap V = \emptyset$ and $U \cap A \neq \emptyset \neq V \cap A$, $A \subset U \cup V$.

Defn A set which is not disconnected is connected.

The only disjoint open sets U, V for which $A \subset U \cup V$ is $U \supset A$ and $V = \emptyset$ (or vice versa).

Thm: A line segment I joining z_0 and z_1 is a connected set in $\mathbb{C}$.



$$f(t) = z_0 t + (1-t) z_1$$

Recall that if $t_0 \in [0,1]$ is s.t. $f(t_0) \in U$ then for suff small δ, $f([t_0-\delta, t_0+\delta] \cap [0,1]) \subset U$. (Same arg. for V).

If $f(0) \in U$ then $\exists$ a set J of the form $[0, t_0)$ s.t. for $t \in J$, $f(t) \in U$. Let $t_0 =$ sup of all such t.