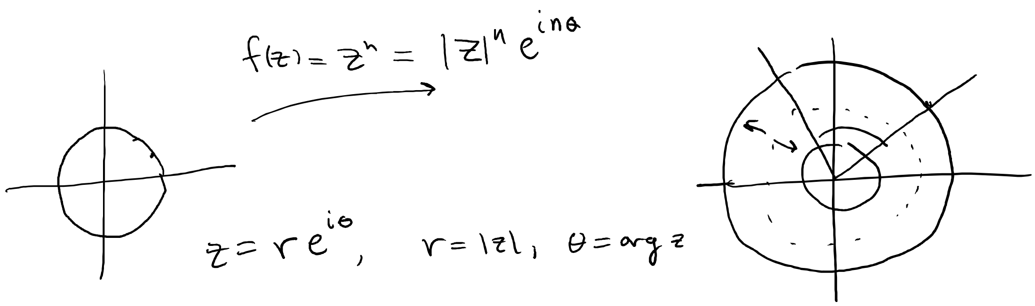
Lec 1/11

Thursday, January 11, 2018 10:19

eg Mapping properties of z^n on ($n \in \mathbb{Z}$.)

(a) circles around the origin

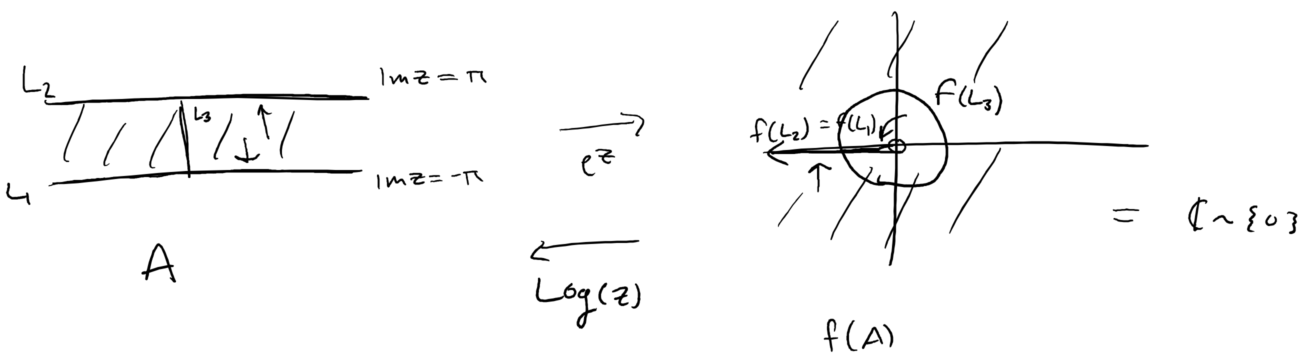
(b) straight lines emanating from the origin.



$z = re^{i\theta}$, $r = |z|$, $\theta = \arg z$

angle is <u>not</u> preserved between two straight lines.

eg univalence of $f(z) = e^z$ in the domain $-\pi < \operatorname{Im} z \le \pi$



e^z is univalent: if $f(z_1) = f(z_2)$ then $e^{z_1 - z_2} = 1$

so $z_1 - z_2 = 2i\pi k$

$k = 0$ by restriction.

so $z_1 = z_2$

univalence works for $\{z : \theta_0 < \operatorname{Im} z \le \theta_0 + 2\pi\}$ for any θ_0.

what is f^{-1}? if $\theta_0 = \pi$ then $f^{-1}(w) = \mathrm{Log}(w) + 2i\pi$.

for general θ_0, $f^{-1}(w) = \ln|w| + i \arg w$

with $\underbrace{\theta_0 < \arg w \leq \theta_0 + 2\pi}_{\text{branch cut}}$

Plane topology

$\mathbb{C} \overset{\text{basically}}{=} \mathbb{R}^2$ w/ euclidean norm.

$z \in \mathbb{C}$, $|z| = \sqrt{x^2 + y^2}$

Notations $\Delta(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ (open ball)

$\bar{\Delta}(z_0, r) = $ " " $\leq$ " " (closed ball)

$\Delta^*(z_0, r) = \{z \in \mathbb{C} : 0 < |z - z_0| < r\}$ (punctured disk)

$K(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$

Defs z_0 is an interior point of $A \subseteq \mathbb{C}$ if $\exists r > 0$ s.t. $\Delta(z_0, r) \subseteq A$.

z_0 is a boundary point of $A \subseteq \mathbb{C}$ if for all $r > 0$ $\Delta(z_0, r) \cap A \neq \emptyset \neq \Delta(z_0, r) \cap (\mathbb{C} \setminus A)$

$A \subseteq \mathbb{C}$ is open if $\forall x \in A$, x is an interior point.

$A \subseteq \mathbb{C}$ is closed if $\mathbb{C} \setminus A$ is open. $\Leftrightarrow$ A contains all of its boundary pts.

$\bar{A} = A \cup \partial A$ boundary of A - set of all boundary pts (closure of A).

Complex sequences $(a_n)_{n=1}^{\infty} \subset \mathbb{C}$. Convergence of (a_n):

Defn: $a_n \to a_0$ if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t. $|a_n - a_0| < \varepsilon$ if $n \geq N$.

Properties: $z_n \to z_0 \iff \operatorname{Re} z_n \to \operatorname{Re} z_0,\ \operatorname{Im} z_n \to \operatorname{Im} z_0$.

if $z_n \to z_0$, $w_n \to w_0$, $z_n \pm w_n \to z_0 \pm w_0$

$c z_n \to c z_0 \quad \forall c \in \mathbb{C}$.

$z_n w_n \to z_0 w_0$

if $w_0 \neq 0$ then $z_n / w_n \to z_0 / w_0$

Def z_0 is an accumulation point of $(z_n)_{n=1}^{\infty}$ if $\forall r > 0$ $\exists k$ s.t. $z_k \in \Delta^*(z_0, r)$. $|\Delta(z_0, r) \cap (z_n)_{n=1}^{\infty}| = \infty$.

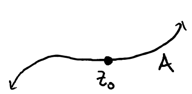
Thm z_0 is an accum. pt. of $(z_n)_{n=1}^{\infty}$ iff $\exists$ subsequence $(z_{n_k})_{k=1}^{\infty}$ for which $z_{n_k} \to z_0$ as $k \to \infty$.

Thm If $z_n \to z_0$ then every subsequence $z_{n_k} \to z_0$

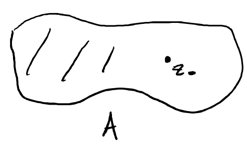
Thm $z_0 \in \bar{A}$ iff $\exists (z_n)_{n=1}^{\infty} \subseteq A$ s.t. $z_n \to z_0$

Thm A is closed iff A contains every acc. pt. of every $(z_n)_{n=1}^{\infty} \subset A$.

Def f is continuous at $z_0 \in A \subseteq \mathbb{C}$ (in the set) if given $\varepsilon > 0$ $\exists\, \delta_\varepsilon > 0$ s.t. $z \in A$ and $|z - z_0| < \delta_\varepsilon \Rightarrow |f(z) - f(z_0)|$



distinction matters



doesn't

does.

equiv: $f(A \cap \Delta(z_0, \delta_\varepsilon)) \subset \Delta(f(z_0), \varepsilon)$

<u>Thm</u>: If $f: A \longrightarrow \mathbb{C}$ and $g: B \longrightarrow \mathbb{C}$, both continuous at $z_0 \in A \cap B$,

then $f \pm g$, $\overset{\forall c \in \mathbb{C}}{cf}$, fg, $\mathrm{Re} f$, $\mathrm{Im} f$, $|f|$, $\bar{f}$ and (if $g(z_0) \neq 0$) $\frac{f}{g}$

are each cts at z_0