Lec 1/10

Wednesday, January 10, 2018 10:18

If $A, B \subseteq \mathbb{C}$

A function $f : A \longrightarrow B$ assigns a value $f(z)$ $\in B$ to each $z \in A$

eg $f(z) = e^z$. Natural domain (largest domain) is $\mathbb{C}$

eg $f(z) = \dfrac{z+1}{z-1}$ N.D.: $\mathbb{C} \setminus \{1\}$ (text uses $S \sim T$ for $S \setminus T$). eg $f(x,y) = x$

eg Rational function (ratl. in z is also ratl. in x and y, reverse is not necessarily true).

$$f(z) = \frac{P_n(z)}{Q_m(z)} = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m} . \ ND: \mathbb{C} \setminus \{ r : r \text{ is a root of } Q_m(z) \}$$

A special subset of functions of a complex variable:

the 'univalent' functions (one-to-one functions)

if $f : A \to \mathbb{C}$ and f is 1-1 $\quad A \longrightarrow f(A) \subset \mathbb{C}$

can create an inverse $f^{-1} : f(A) \longrightarrow A$.

Remark: in our context univalence will refer to map $A \to f(A)$, a bijection.

Sometimes it is helpful to separate real & imaginary parts of $f(z)$.

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y) \qquad (u = \operatorname{Re} f(z),\ v = \operatorname{Im} f(z)).$$

eg: $f(z) = z^3$. determine u & v.

$$(x+iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3$$

so $u(x,y) = x^3 - 3xy^2$, $v(x,y) = 3x^2y - y^3$.

eg $f(z) = e^z$. $\quad e^{x+iy} = e^x(\cos y + i\sin y)$

$$= \underbrace{(e^x\cos y)}_{u} + i\,\underbrace{(e^x \sin y)}_{v}$$

Combining functions:

1) cf for $c \in \mathbb{C}$. $(cf)(z) = c(f(z))$

2) $(f+g)(z) = f(z) + g(z)$

3) $(fg)(z) = f(z)g(z)$

4) $(f\circ g)(z) = f(g(z))$

$f: A \to \mathbb{C}$, $g: B \to \mathbb{C}$

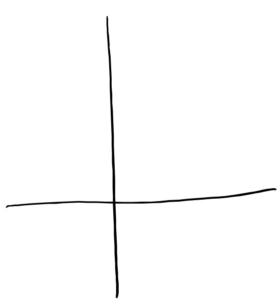
$f+g, fg : A \cap B \to \mathbb{C}$

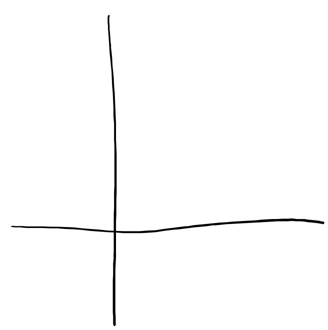
$f\circ g : B \cap g^{-1}(A) \longrightarrow \mathbb{C}$

eg $f(z) = e^z$, $g(z) = z^2$, $(f\circ g)(z) = e^{z^2}$. $\quad$ Z $\quad$ 7 $\quad$ 123456 789

ex: $f(z) = az + b$, $a, b \in \mathbb{C}$.

Determine the mapping properties of f.





$a = |a|e^{i\theta}$ $\qquad$ $\theta = \arg a$

<u>Def</u> $g(z) = e^{i\theta} z$ g is a rotation by θ.

$h(z) = |a| z$ h is a scale by $|a|$.
(a dilation).

$K(z) = z + b$ shift by b (translation)

$f = K \circ h \circ g$

$f(z) = \left(|a|\left(e^{i\theta} z\right)\right) + b$

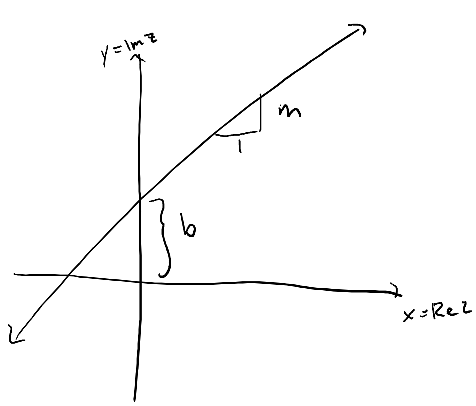
$g(\diagdown) =$

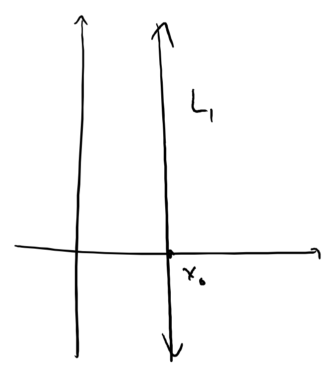
$h(\diagup) =$

$K(\diagup) =$

eg: Describe the image of arbitrary line $L = \{z = x+iy : ax+by = c\}$

under the mapping $f(z) = e^z$

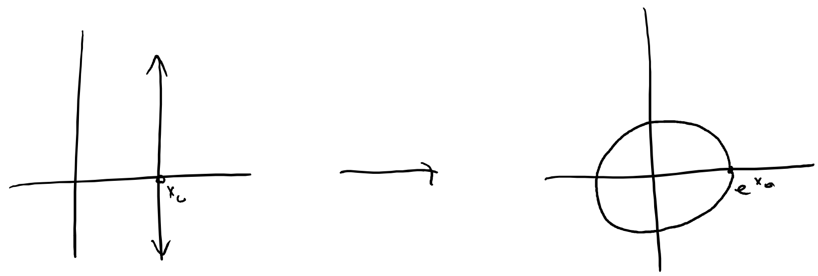




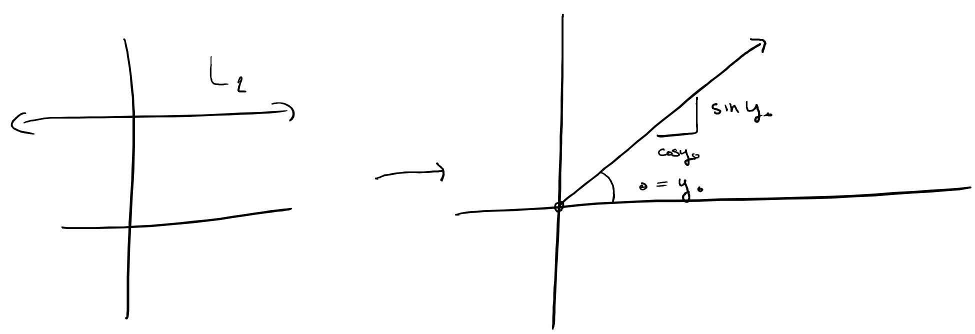
Special case: $L_1 = \{x = x_0\}$

on L_1, $z = x_0 + iy$. So $e^z = e^{x_0}(\cos y + i \sin y)$

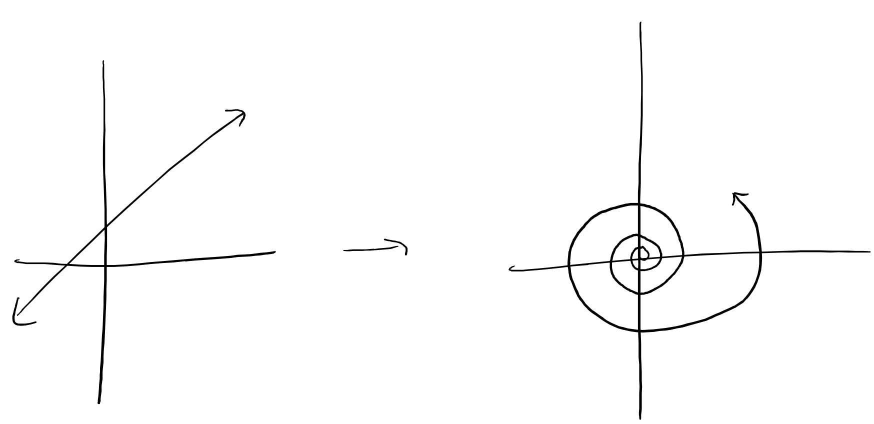
So



$L_2 = \{y = y_0\}$ $\quad e^z = e^x(\cos y_0 + i \sin y_0)$



on L: $f(z) = e^x e^{iy} = e^x e^{i(mx+b)}$



$r = e^x, \qquad x = \ln r$

$\theta = mx + b = m \ln r + b \Rightarrow e^{\left(\frac{\theta - b}{m}\right)} = r$

a logarithmic spiral.

eg Show that $f(z) = \dfrac{1-z}{1+z}$

maps $D = \{z : |z| < 1\}$ into $H = \{w : \operatorname{Re} w > 0\}$

$$f(z) = \frac{(1-z)\,(1+\bar{z})}{(1+z)\,(1+\bar{z})} = \frac{1 - |z|^2 - (z - \bar{z})}{|1+z|^2}$$

$$u = \frac{1-|z|^2}{|1+z|^2} \qquad v = \frac{-(z-\bar{z})}{i\,|1+z|^2}$$

indeed $u > 0$ for $|z| < 1$.

$f : D \longrightarrow H$

Does f surjective? is f univalent?

Try solve $w = \dfrac{1-z}{1+z}$ for $z \in D$, given $w \in H$

$\Rightarrow z = \dfrac{1-w}{1+w}$ is $z \in D$?

$$|z| = \left| \frac{(1-w)}{(1+w)} \frac{(1+\bar{w})}{(1+\bar{w})} \right| = \frac{(1-|w|^2) - (w - \bar{w})}{|1+w|^2}$$

$$|z| = \left| \frac{(1-w)}{(1+w)} \frac{(1+w)}{(1+\bar{w})} \right| = \frac{\quad}{|1+w|^2}$$

Check that $$\frac{\left(1-|w|^2\right)^2 + \left(\frac{w-\bar{w}}{i}\right)^2}{|1+w|^4} < 1$$