Lec 11/1

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<u>Lemma</u>: let φ be a solution to $L(y)=0$. Define

$$u(x) = |\varphi(x)|^2 + |\varphi'(x)|^2 + \cdots + |\varphi^{(n-1)}(x)|^2 =$$

Then $|u(x)| \leq e^{2k|x-x_0|} u(x_0)$, $k = 1 + |a_0| + \cdots |a_n|$

<u>Proof</u>: $u(x) = \varphi(x)\overline{\varphi(x)} + \cdots + \varphi^{(n-1)}(x)\overline{\varphi^{(n-1)}(x)}$

$$u'(x) = \varphi'(x)\overline{\varphi(x)} + \varphi(x)\overline{\varphi'(x)} + \cdots + \varphi^{(n)}(x)\overline{\varphi^{(n-1)}(x)} + \varphi^{(n-1)}(x)\overline{\varphi^{(n)}(x)}$$

$$|u'(x)| \leq 2\left(|\varphi'(x)\varphi(x)| + \cdots + |\varphi^{(n)}(x)||\varphi^{(n-1)}(x)|\right)$$

and $|\varphi^{(n)}(x)| \leq |a_1||\varphi^{(n-1)}(x)| + \cdots + |a_n||\varphi(x)|$

so, using algebra, we get

$$|u'(x)| \leq 2k|u(x)|$$

note u is positive

so $-2ku(x) \leq u'(x) \leq 2k\, u(x)$

so $u'(x) + 2ku(x) \geq 0$ $\quad u'(x) - 2ku(x) \leq 0$

so $(e^{2kx}u(x))' \geq 0$, $(e^{-2kx}u(x)) \leq 0$

from here, the proof was given in a previous class.

<u>Corollaries</u>

1: Suppose $L(\varphi) = 0$, $\varphi(x_0) = \varphi'(x_0) = \cdots = \varphi^{(n-1)}(x_0) = 0$

then $\varphi = 0$.

2: suppose $L(\varphi) = L(\psi) = 0$, $\varphi(x_0) = \psi(x_0), \ldots, \varphi^{(n-1)}(x_0) = \psi^{(n-1)}(x_0)$

then $\varphi = \psi$

Now we can prove that

3: if $\varphi_1, \dots, \varphi_n$ are solutions to $L(y) = 0$ and are linearly independent, The Wronskian of these solutions is never 0.

Proof: Suppose $W(\varphi_1, \dots, \varphi_n)(x_0) = 0$. then $\exists\, c_1, \dots, c_n$ not all 0 s.t.

$$c_1 \varphi_1(x_0) + \cdots + c_n \varphi_n(x_0) = 0$$

$$c_1 \varphi_1'(x_0) + \cdots + c_n \varphi_n'(x_0) = 0$$

$$\vdots$$

$$c_1 \varphi^{(n-1)}(x_0) + \cdots + c_n \varphi^{(n-1)}(x_0) = 0$$

take $\varphi(x) = c_1 \varphi_1(x) + \cdots + c_n \varphi_n(x)$

then φ satisfies trivial initial cond, so $\varphi = 0$.

but then $\{\varphi_1, \dots, \varphi_n\}$ are linearly dep. $\times$

So $W(x) = 0 \;\forall x$.

$L(y)=0,\; y(x_0) = \alpha_0, \dots, y^{(n-1)}(x_0) = \alpha_{n-1}$

Thm (1) every initial value problem has exactly 1 solution.
(2) $\dim N(L) = n$.

The existence of the solution is given by the fact that $W(\varphi_1, \dots, \varphi_n)(x_0) \neq 0$ for any x_0.

Any solution φ satisfies some initial value problem, so $\exists$ a liner combination of φ_i which is φ, so

$\{\varphi_1, \dots, \varphi_n\}$ are a basis for $N(L)$ so $\dim N(L) = n$.

If a_i are all real, then complex roots come in pairs $\sigma \pm i\tau$, so some sols are $e^{\sigma x} \sin \tau x$ and $e^{\sigma x} \cos \tau x$.