Lec 4/7

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<u>Stokes's Theorem</u> let $S \subseteq \mathbb{R}^3$ be ccosswb. Let $\vec{F}: U \to \mathbb{R}^3$ be a vector field defined on U open $\subseteq S$. Then

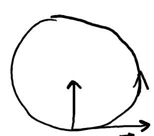
$$\int_{\partial S} \vec{F} \cdot d\vec{x} = \iint_S \operatorname{curl}(\vec{F}) \cdot \vec{n}\, dA$$

where ∂S is oriented so $\vec{n} \times$ (tangent to ∂S) points into S.

$\vec{n} \times (\text{tangent to } \partial S) \perp \vec{n} \Rightarrow$ tangent to S at ∂S

Green's theorem is a special case, $S \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3$

$\vec{n} = \vec{k}$ points up



$\vec{i}$ = tangent vector, $\vec{k} \times \vec{i} = \vec{j}$

$F = P\vec{i} + Q\vec{j}$ $\qquad \operatorname{curl}(\vec{F}) = \text{something} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$

$$\iint_S \operatorname{curl}(\vec{F}) \cdot \vec{n}\, dA = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

<u>Proof</u>: If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

it suffices to prove ST for $\vec{F} = P\vec{i}$, $Q\vec{j}$, $R\vec{k}$ separately

suffices to prove ST for $\vec{F} = P\vec{i}$

$\operatorname{curl} \vec{F} = \frac{\partial P}{\partial z}\vec{j} - \frac{\partial P}{\partial y}\vec{k}$

Lets assume S parametrized by $\vec{G}: W \to S \subseteq \mathbb{R}^3$, $W \subseteq \mathbb{R}^2$

and assume $\vec{G}$ is C^2, and $\vec{G}(\partial W) = \partial S$

assume $\partial_u \vec{G} \times \partial_v \vec{G}$ points in direction of $\vec{n}$

$\vec{x} = \vec{G}(\vec{u})$

$$\vec{G}_u \times \vec{G}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_u x & \partial_u y & \partial_u z \\ \partial_v x & \partial_v y & \partial_v z \end{vmatrix} = \frac{\partial(y,z)}{\partial(u,v)} \vec{i} - \frac{\partial(x,z)}{\partial(u,v)} \vec{j} + \frac{\partial(x,y)}{\partial(u,v)} \vec{k}$$

$$\iint_S \operatorname{curl}(\vec{F}) \cdot \vec{n} \, dA = \iint_S \left(\frac{\partial P}{\partial z} \vec{j} - \frac{\partial P}{\partial y} \vec{k} \right) \cdot \left(\vec{G}_u \times \vec{G}_v \right) dA$$

$$= \iint_W \left(\frac{\partial P}{\partial z} \frac{\partial(z,x)}{\partial(u,v)} - \frac{\partial P}{\partial y} \frac{\partial(x,y)}{\partial(u,v)} \right) du \, dv$$

for simplicity, assume $\partial W, \partial S$ consist of a $\overset{\text{simple}}{\vee}$single closed curve

Parametrize it by $\vec{u} : [a,b] \to \partial W \subseteq \mathbb{R}^2$

$\vec{u}(t) = (u(t), v(t)) \Rightarrow \vec{G}(\vec{u}(t)) \quad a \le t \le b$ params ∂S.

note $\vec{x}(t) = (x(\vec{u}(t)), y(\vec{u}(t)), z(\vec{u}(t)))$

$$\int_{\partial S} \vec{F} \cdot d\vec{x} = \int_a^b P\vec{i} \cdot \frac{d\vec{x}}{dt} \, dt = \int_a^b P \frac{dx(\vec{u}(t))}{dt} \, dt$$

chain rule $= \displaystyle\int_a^b P \left(\frac{\partial x}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial x}{\partial v} \frac{\partial v}{\partial t} \right) dt$

$$= \int_a^b \left(P \frac{\partial x}{\partial u} \vec{i} + P \frac{\partial x}{\partial v} \vec{j} \right) \cdot \overbrace{\left(\frac{du}{dt} \vec{i} + \frac{dv}{dt} \vec{j} \right)}^{= \frac{d\vec{u}}{dt}} dt$$

$$= \int_{\partial W} \left(P \frac{\partial x}{\partial u} \vec{i} + P \frac{\partial x}{\partial v} \vec{j} \right) \cdot d\vec{u}$$

Green's thm $= \displaystyle\iint_W \left(\frac{\partial}{\partial u} \left(P \frac{\partial x}{\partial v} \right) - \frac{\partial}{\partial v} \left(P \frac{\partial x}{\partial u} \right) \right) du \, dv$

Now:

$$\frac{\partial}{\partial u}\left(P\frac{\partial x}{\partial v}\right) = \left(\cancel{\frac{\partial P}{\partial x}\frac{\partial x}{\partial u}} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial u}\right)\frac{\partial x}{\partial v} + \cancel{P\left(\frac{\partial^2 x}{\partial u \partial v}\right)} \qquad C^2 \text{ condition}$$

$$+$$

$$-\frac{\partial}{\partial v}\left(P\left(\frac{\partial x}{\partial u}\right)\right) = -\left(\cancel{\frac{\partial P}{\partial x}\frac{\partial x}{\partial v}} + \frac{\partial P}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial v}\right)\frac{\partial x}{\partial u} - \cancel{P\left(\frac{\partial^2 x}{\partial v \partial u}\right)}$$

$$=$$

$$\hookrightarrow \frac{\partial P}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) - \frac{\partial P}{\partial y}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)$$

$$= \frac{\partial P}{\partial z}\frac{\partial(z,x)}{\partial(u,v)} - \frac{\partial P}{\partial y}\frac{\partial(x,y)}{\partial(u,v)}$$

So the two sides are equal $\square$

Note: An unknotted simple closed curve in $\mathbb{R}^3$ is the boundary of infinitely many surfaces.

Sometimes useful trick for computing a line integral

$$\int_C \vec{F}\cdot dx = \iint_S \operatorname{curl}(\vec{F})\cdot\vec{n}\, dA$$ (perhaps find something that makes this simple.