Lec 4/6

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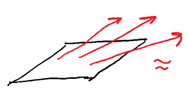
# Physical Interpretation of Surface Integrals

substance flowing in $\mathbb{R}^3$, how much flows through $S$, a smooth orientable surface

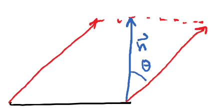
flow described by a vector field $\vec{J}(x,y,z,t) = \underbrace{\rho(x,y,z,t)}_{\text{density of material}}\underbrace{\vec{v}(x,y,z,t)}_{\text{velocity of material}}$

**Proposition:** $\iint_S \vec{J}\cdot\vec{n}\, dA$ = rate of flow through surface at time $t$.

**Proof sketch:** As in derivation of surface area, approximate $S$ by parallelograms formed by tangent lines at various points.



substance flowing through approximating parallelogram $\approx \vec{v}\,\Delta t$




(side view of parallelepiped)

Volume of ppd = Area of base · height

$\Rightarrow$ The mass of flow through surface during $\Delta t$ $\approx \sum_{j=1}^{n}\sum_{i=1}^{m} \rho_{ij} A_{ij}\, \vec{v}\Delta t \cdot \vec{n}_u$

as "things get smaller", we get $\int_S \rho\, \vec{v}\cdot\vec{n}\, dA\, \Delta t = \left(\iint_S \vec{J}\cdot\vec{n}\, dA\right)\underset{dt}{\underset{\downarrow}{\Delta t}} = \underset{dM}{\underset{\downarrow}{\Delta M}}$

So $\frac{dM}{dt}$ = rate of flow = $\iint_S \vec{J}\cdot\vec{n}\, dA$ ■

**Corollary** (Conservation of mass): $\frac{\partial\rho}{\partial t} + \operatorname{div}(\vec{J}) = 0$

**Proof:** if $V$ is a region in $\mathbb{R}^3$ enclosed by a compact connected orientable surface without internal boundary, then $\iiint \rho(x,y,z,t)\, dV$ = total mass in $V$ at time $t$

then $\displaystyle\iiint_{\mathcal{V}} \rho(x,y,z,t)\, dV$ = total mass in $\mathcal{V}$ at time $t$.

$$\frac{d}{dt}\iiint \rho(x,y,z,t)\, \partial V = \text{rate at which mass enters } \mathcal{V}.$$

$$= \text{rate at which mass enters thru } \partial \mathcal{V}$$

$\|$

outward normal

$$\iiint_{\mathcal{V}} \partial t \quad \partial V \qquad = -\iint_{\partial \mathcal{V}} \vec{J}\cdot\vec{n}\, \partial A$$

$$= -\iiint_{\mathcal{V}} \operatorname{div}\vec{J}\, \partial V$$

$$\Rightarrow \iiint_{\mathcal{V}} \qquad\qquad \int$$

$$u(\vec{x}) = \iiint_{\mathbb{R}^3} \text{——} \qquad\qquad \iint_{\mathbb{R}^3} \frac{\rho(\vec{x}+\vec{y})}{|\vec{y}|}\, \partial^3 \vec{y}$$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}$$

**Theorem 5.46** suppose $\rho$ is $C^2$ and $u(\vec{x}) = \displaystyle\iiint_{\mathbb{R}^3} \frac{\rho(\vec{x}+\vec{y})}{|\vec{y}|}\, \partial^3\vec{y}$. Then $\nabla^2 u(\vec{x}) = -4\pi\rho(\vec{x})$.

Proof:

$$\nabla^2 u(\vec{x}) = \iiint_{\mathbb{R}^3} \frac{\nabla_{\vec{x}}^2 \rho(\vec{x}+\vec{y})}{|\vec{y}|}\, \partial^3\vec{y} = \iiint_{\mathbb{R}^3} \frac{\nabla_{\vec{y}}^2 \rho(\vec{x}+\vec{y})}{|\vec{y}|}\, \partial^3\vec{y}$$

$$\iiint_{\mathbb{R}^3} \qquad\qquad \iiint_{\mathbb{R}^3} \frac{\quad}{|\vec{y}|}$$

$$\partial_i \rho(\underbrace{\vec{x}+\vec{y}}_{\vec{z}}) = \frac{\partial}{\partial z_i} \rho(\vec{x}+\vec{y}) \underbrace{\frac{\partial z_i}{\partial x_i}}_{=1} = \frac{\partial}{\partial y_i} \rho(\vec{x}+\vec{y})$$

now apply green's formulas:

$$\nabla^2 f \; g$$
$$= (\nabla^2 \rho)(|\vec{y}|^{-1})$$

$$\iiint_{\mathbb{R}^3} \frac{\nabla_{\vec{y}}^2 \rho(\vec{x}+\vec{y})}{|\vec{y}|} \, \partial^3 \vec{y} = \iiint_{R_{\varepsilon,k}} \frac{\nabla_{\vec{y}}^2 \rho(\vec{x}+\vec{y})}{|\vec{y}|} \, \partial^3 \vec{y}$$

where $R_{\varepsilon,k} \subseteq \mathbb{R}^3$ is where

where $\rho(\vec{x}+\vec{y}) = 0$ outside of $B(k, \vec{0})$

$$\text{(green's formulas)} \quad = \lim_{\varepsilon \to 0} \Bigg( \iiint_{R_{\varepsilon,k}} \rho(\vec{x}+\vec{y}) \, \nabla_{\vec{y}}^2 (|\vec{y}|^{-1}) \, \partial^3 \vec{y}$$
$$+ \iint_{\partial R_{\varepsilon,k}} \nabla_{\vec{y}} \rho(\vec{x}+\vec{y}) \, |y|^{-1} \cdot \vec{n} \, \partial A$$
$$- \iint_{\partial R_{\varepsilon,k}} \rho(\vec{x}+\vec{y}) \, \nabla_{\vec{y}} (|\vec{y}|^{-1}) \cdot \vec{n} \, \partial A \Bigg)$$

Note

$$\nabla(|\vec{y}|^{-1}) = \frac{-\vec{y}}{|\vec{y}|^2}$$

$$\nabla^2(|y|^{-1}) = \operatorname{div}\left(\frac{-\vec{y}}{|\vec{y}|^2}\right)$$
$$= 0 \text{ for } \vec{y} \neq 0$$

$$\partial R_{\varepsilon,k} = \text{Sphere of rad } \varepsilon \; \cup \; \text{Sphere of rad } k$$