Lec 4/5

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Divergence Theorem

Let $\mathcal{V} \subseteq \mathbb{R}^3$ be bounded by a compact connected orientable surface w/o boundary S_0 on the outside and CCOSWOBs $S_1, \dots, S_k$ on the inside.

Let $\vec{F} : U \to \mathbb{R}^3$, $U \supseteq \mathcal{V}$ be C^1. Then

$$\iiint_{\mathcal{V}} \operatorname{div} \vec{F} \, \partial V = \iint_{\partial \mathcal{V}} \vec{F} \cdot \vec{n} \, \partial A = \sum_{j=0}^{k} \iint_{S_j} \vec{F} \cdot \vec{n}_j \, \partial A$$

where $\vec{n}_0$ points out & $\vec{n}_{j>0}$ point in. (in $\mathbb{R}^3$)

(all $\vec{n}_j$ point out of $\mathcal{V}$)

Proof (ish)

* Reduce to case where $\mathcal{V}$ is bounded by a single surface.
i.e. if $\mathcal{V}_j$ is the region bounded by S_j for $j = 0, \dots, k$

then
$$\iiint_{\mathcal{V}} \operatorname{div} \vec{F} \, \partial V = \iiint_{\mathcal{V}_0} \operatorname{div} \vec{F} \, \partial V - \sum_{j=1}^{k} \iiint_{\mathcal{V}_j} \operatorname{div} \vec{F} \, \partial V$$

** We'll prove the divergence theorem for regions which can be decomposed in 3 diff ways:

(1) Finite union of xy-simple regions → $(x,y,z) \in S \Leftrightarrow (x,y) \in R \subseteq \mathbb{R}^2$ (bounded by piecewise C^1 curve), $\varphi(x,y) \le z \le \psi(x,y)$

(2) " " xz-simple " "

(3) " " yz-simple " "

(Folland requires that $\mathcal{V}$ be decomposed into finitely many regions which are simultaneously xy, xz, yz-simple).

If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, it suffices to show that

(1) $$\iiint_{\mathcal{V}} \frac{\partial R}{\partial z} \partial V = \iint_{\partial \mathcal{V}} R\vec{k} \cdot \vec{n} \, \partial A$$

(2) $$\iiint_{\mathcal{V}} \frac{\partial Q}{\partial y} \partial V = \iint_{\partial \mathcal{V}} Q\vec{j} \cdot \vec{n} \, \partial A$$

(3) $$\iiint_{\mathcal{V}} \frac{\partial P}{\partial x} \partial V = \iint_{\partial \mathcal{V}} P\vec{i} \cdot \vec{n} \, \partial A$$

adding these together gives

$$\iiint_{\mathcal{V}} \operatorname{div} \vec{F} \, \partial V = \iint_{\partial \mathcal{V}} \vec{F} \cdot \vec{n} \, \partial A$$

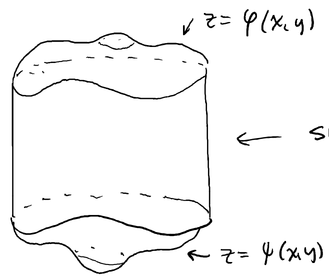
Prove ① by decomposing $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \cdots \cup \mathcal{V}_m$ where $\mathcal{V}_j$ xy-simple and $\mathcal{V}_i \cap \mathcal{V}_j$ is a surface contained inside $\mathcal{V}$ (has 3-d measure 0).

$$\iiint_{\mathcal{V}} \frac{\partial R}{\partial z}\, dV = \sum_{j=1}^{m} \iiint_{\mathcal{V}_j} \frac{\partial R}{\partial z}\, dV \overset{?}{=} \sum_{j=1}^{m} \iint_{\partial \mathcal{V}_j} R\vec{k}\cdot\vec{n}_j\, dA = \iint_{\partial \mathcal{V}} R\vec{k}\cdot\vec{n}\, dA$$

↓ this is true by cancellation on internal boundaries.

(for ②, ③, use xz-simple, yz-simple)

Assume V xy-simple. want to show that $\iiint_{\mathcal{V}} \frac{\partial R}{\partial z}\, dV = \iint_{\partial \mathcal{V}} R\vec{k}\cdot\vec{n}\, dA$



← side is a generalized cylinder described by $g_i(x,y) = 0$ C^1 eqns $i = 1, 2, \ldots, k$. base of cylinder is $\mathcal{R}$

$$\iiint_{V} \frac{\partial R}{\partial z}\, dV = \iint_{\mathcal{R}} \int_{\psi(x,y)}^{\varphi(x,y)} \frac{\partial R}{\partial z}\, dz\, dA$$

$$= \iint_{\mathcal{R}} R(x,y,z)\Big|_{\psi(x,y)}^{\varphi(x,y)}\, dA$$

$$= \iint_{\mathcal{R}} R(x,y,\varphi(x,y))\, dA - \iint_{\mathcal{R}} R(x,y,\psi(x,y))\, dA$$

$$\iint_{\partial V} R\vec{k}\cdot\vec{n}\, dA = \iint_{\text{top of } V} R\vec{k}\cdot\vec{n}\, dA + \iint_{\text{bottom of } V} R\vec{k}\cdot\vec{n}\, dA + 0$$

↙ $\vec{k} \perp \vec{n}$ on sides.

$$\begin{cases} \vec{G}_{top}(x,y) = (x,y,\varphi(x,y)) & \vec{n}_{top} = -\varphi_x(x,y)\vec{i} - \varphi_y(x,y)\vec{j} + \vec{k} \\ \vec{G}_{bot}(y,x) = (x,y,\psi(x,y)) & \vec{n}_{bot} = \psi_x(x,y)\vec{i} + \psi_y(x,y)\vec{j} - \vec{k} \\ R\vec{k}\cdot\vec{n}_{top} = R(x,y,\varphi(x,y)) & \\ R\vec{k}\cdot\vec{n}_{bot} = -R(x,y,\psi(x,y)) & \end{cases}$$

$$R\vec{k}\cdot n_{bot} = -R(x,y,\varphi(x,y))$$

$$= \iint_R R(x,y,\varphi(x,y))\,dA - \iint_R R(x,y,\varphi(x,y))\,dA$$

So the sides of the equation are equal. This proves the theorem ●

Green's Formulas

$$\iint_{\partial V} f\nabla g\cdot\vec{n}\,dA = \iiint_V (\nabla f\cdot\nabla g + f\nabla^2 g)\,dV$$

$$\iint_{\partial V} (f\nabla g - g\nabla f)\cdot\vec{n}\,dA = \iiint_V (f\nabla^2 g - g\nabla^2 f)\,dV$$

Proof

$$\operatorname{div}(f\nabla g) = \operatorname{div}\left(f\frac{\partial g}{\partial x}\vec{i} + f\frac{\partial g}{\partial y}\vec{j} + f\frac{\partial g}{\partial z}\vec{k}\right)$$

$$= \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \cdots$$

$$+ f\frac{\partial^2 g}{\partial x^2} + \cdots$$

$$= \nabla f\cdot\nabla g + f\nabla^2 g$$