Lec 4/4

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Vector Derivatives

Notation $\nabla = (\partial_1 -, \partial_2 -, \ldots, \partial_n -)$

if $f : U^{\subseteq \mathbb{R}^n} \to \mathbb{R}$ then $\nabla f = (\partial_1 f, \partial_2 f, \ldots, \partial_n f) = \text{grad } f$

if $\vec{F} : U^{\subseteq \mathbb{R}^n} \to \mathbb{R}^n$ then $\nabla \cdot \vec{F} = \partial_1 F_1 + \partial_2 F_2 + \cdots + \partial_n F_n = \text{div } \vec{F}$

if $\vec{F} : U^{\subseteq \mathbb{R}^3} \to \mathbb{R}^3$ then
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix} = (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1) = \text{curl } \vec{F}$$

$$\begin{tikzcd} \text{scalar} \arrow[r, "\text{grad}"] \arrow[rr, bend left, "0"] & \text{vector} \arrow[r, "\text{curl}"] \arrow[rr, bend right, "0"'] & \text{vector} \arrow[r, "\text{div}"] & \text{scalar} \end{tikzcd}$$

0's follow from mixed partials.

$(\text{div} \circ \text{grad}) f = \partial_1^2 f + \partial_2^2 f + \cdots + \partial_n^2 f = \nabla^2 f$ = Laplacian.

If $\nabla^2 f = 0$ then f is a harmonic function.

$$\begin{array}{ccc} x+yi & \mapsto & u+vi \\ g : \mathbb{C} & \longrightarrow & \mathbb{C} \\ \| & & \| \\ \mathbb{R}^2 & & \mathbb{R}^2 \\ (x,y) & \mapsto & (u,v) \end{array}$$

differentiable in complex sense (analytic)

$g(x,y) = (u(x,y), v(x,y))$ then u, v harmonic.

Surface Integrals

$$\iint_S \vec{F} \cdot \vec{n} \, dA$$

S described by $\vec{G} : R^{\subseteq \mathbb{R}^2} \longrightarrow \mathbb{R}^3$ of class C^1.

2-D

More generally $S = S_1 \cup S_2 \cup \cdots \cup S_k$ which intersect in measure zero

So $\iint_S \vec{F} \cdot \vec{n} \, dA = \sum_{j=1}^{k} \iint_{S_j} \vec{F} \cdot \vec{n} \, dA$

Definition We say two topological spaces $(X, \mathcal{T}_1)$, $(Y, \mathcal{T}_2)$ are homeomorphic if there is a 1-1 onto function $f: X \to Y$ s.t. f and f^{-1} are continuous everywhere.

Definition A (Hausdorff) topological space $(X, \mathcal{T})$ is called a surface w/o boundary if for each $x_0 \in X$ there is a neighborhood U of x_0 and a homeomorphism $U \rightleftarrows B(r, \vec{0}) \subseteq \mathbb{R}^2$. ← 2→n / surface → n-dim manifold

A surface with a boundary is a (Hausdorff) Topological space $(X, \mathcal{T})$ s.t. for any point $x_0 \in X$ there is an open $U \ni x_0$ s.t.

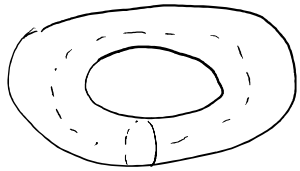
either ① $U \leftrightarrows B(r, \vec{0}) \subseteq \mathbb{R}^2$

or ② $U \rightleftarrows \{(x,y) : x^2 + y^2 < r, \ y \geq 0\}$

(points of type ② are "intrinsic boundary points")

Classification of surfaces

(*) Any compact, connected, orientable surface w/o boundary is homeomorphic to one of the following:



Parametrized by $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq 2\pi$

Analog of jordan Curve theorem

If X, a subset of $\mathbb{R}^3$, is a surface of type (*)

then $\mathbb{R}^3 \setminus X$ consists of two disjoint open components:

1. a bounded component (inside)
2. an unbounded component (outside)

Divergence Theorem

If $\mathcal{V}$ is a region in $\mathbb{R}^3$ which is inside a compact connected orientable surface w/o boundary S_0 and outside surfaces $S_1, \dots, S_K$, And $\vec{F}: \overset{\mathcal{V}\,\subset}{\mathcal{U}} \to \mathbb{R}^3$

then $$\sum_{j=0}^{K} \iint_{S_j} \vec{F} \cdot \vec{n}_j \, dA = \iiint_{\mathcal{V}} \operatorname{div} \vec{F} \, dV$$

note $\vec{n}_0$ points out of S_0 and $\vec{n}_{j>0}$ points inside $S_{j\geq 1}$


$\vec{n}_1$
S_1
$\vec{n}_0$
S_0