Lec 4/3

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Surface area formula

$\vec{G} : \mathcal{R}^{\subseteq \mathbb{R}^2} \to \mathbb{R}^3, \quad S = \vec{G}(\mathcal{R}), \quad A(S) = \iint_{\mathcal{R}} |\vec{G}_u \times \vec{G}_v| \, dA$ in $\mathbb{R}^3$

if $3 \to n$, we use $\iint_{\mathcal{R}} \sqrt{|\vec{G}_u|^2 |\vec{G}_v|^2 - (\vec{G}_u \cdot \vec{G}_v)^2} \, dA$

$$\int_C f \, ds = \int_a^b f(\vec{g}(t)) \, |\vec{g}'(t)| \, dt \qquad \text{if } C = \vec{g}([a,b]) \subseteq \mathbb{R}^n, \quad f : \underset{\substack{\cup\!\!\mid \\ C}}{U} \to \mathbb{R}$$

(like integral of scalar).

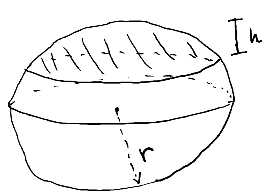
surface integral of a scalar:

$$\iint_S f \, dA = \iint_{\mathcal{R}} f(\vec{G}(u,v)) \, |\vec{G}_u \times \vec{G}_v| \, du\,dv$$

or $\sqrt{|\vec{G}_u|^2 |\vec{G}_v|^2 - (\vec{G}_u \cdot \vec{G}_v)^2}$

where $S = \vec{G}(\mathcal{R})$, $\mathcal{R} \subseteq \mathbb{R}^2$, $S \subseteq \mathbb{R}^{3 \to n}$, and $f : U \to \mathbb{R}$, $U \supseteq S$

<u>Example</u> Calculate the centroid of a spherical cap of spherical radius r, height h.



by symmetry, it lies on z-axis $\bar{x} = \bar{y} = 0$.

$$\bar{z} = \frac{\iint_S z \, dA}{\iint_S dA}$$

Use spherical coords: $x = r\sin\varphi\cos\theta \quad y = r\sin\varphi\sin\theta \quad z = r\cos\varphi$.

so $S = (x,y,z)$, $\quad 0 \le \theta \le 2\pi \quad 0 \le \varphi \le \alpha \longrightarrow$ where $z = r - h \Rightarrow \cos\varphi = 1 - \frac{h}{r}$ so $\alpha = \arccos\left(1 - \frac{h}{r}\right)$

$\vec{G}_\varphi = (r\cos\varphi\cos\theta, \ r\cos\varphi\sin\theta, \ -r\sin\varphi) \qquad \vec{G}_\theta = (-r\sin\varphi\sin\theta, \ r\sin\varphi\cos\theta, \ 0)$

$|\vec{G}_\varphi \times \vec{G}_\theta| = r^2 \sin\varphi \qquad$ so $dA = r^2 \sin\varphi \, d\varphi \, d\theta$

$$\bar{z} = \frac{\int_0^{2\pi}\int_0^{\alpha} r\cos\varphi \; r^2 \sin\varphi \, d\varphi \, d\theta}{\int_0^{2\pi}\int_0^{\alpha} r^2 \sin\varphi \, d\varphi \, d\theta} = \frac{\frac{r}{2}\int_0^{\alpha} \sin(2\varphi)\, d\varphi}{1-\cos(\alpha)}$$

$$= \frac{\frac{r}{2}\left[\frac{-\cos(2\alpha)}{2} + \frac{\cos(0)}{2}\right]}{\frac{h}{r}} = \frac{r^2}{4h}\left[1 - \left[\cos^2(\alpha) + \sin^2(\alpha)\right]\right]$$

$$= \frac{r^2}{4h} - \frac{r^2}{4h}\left(\frac{h^2}{r^2} + 1 - \frac{h^2}{r^2}\right)$$

$$= r - \frac{h}{2} \, ?$$

Line Integral of a vector field

$$\int_C \vec{F} \cdot d\vec{x} = \int_a^b \vec{F}(\vec{g}(t)) \cdot \vec{g}\,'(t)\, dt \quad \text{where } \mathbb{R}^n \supseteq C = \vec{g}([a,b]) \text{ and } \vec{F} : U \to \mathbb{R}^n, \; U \supseteq C$$

def $= \int_C \vec{F} \cdot \vec{T} \, ds$ ← unit tangent

$$= \int \vec{F}(\vec{g}(t)) \frac{\vec{g}\,'(t)}{|\vec{g}\,'(t)|} |\vec{g}\,'(t)| \, dt$$

Analog for surface integrals.

$S = \vec{G}(R)$, $R \subseteq \mathbb{R}^2$, $S \subseteq \mathbb{R}^3$ $\quad \vec{F} : U \to \mathbb{R}^3$, $U \supseteq S$ $\quad$ but $\iint_S \vec{F} \cdot \vec{T} \, dA$ doesn't make sense: no unique unit tangent to S.

but there is a unique unit normal to S at a point up to ± 1.

(the $\pm$ determines the "orientation" of the curve)

$\iint_S \vec{F} \cdot \vec{n} \, dA$ makes sense for orientable surfaces.

fact: Closed surfaces (surfaces w/o boundary) are orientable.

note $\vec{G}_u, \vec{G}_v$ tangent to surface, so $\vec{G}_u \times \vec{G}_v$ perpendicular.

so $\vec{n} = \pm \dfrac{\vec{G}_u \times \vec{G}_v}{|\vec{G}_u \times \vec{G}_v|}$, $dA = |\vec{G}_u \times \vec{G}_v| \, du \, dv$

$$\iint_S \vec{F} \cdot \vec{n} \, dA = \pm \iint_R \vec{F}(\vec{G}(u,v)) \cdot (\vec{G}_u \times \vec{G}_v) \, du \, dv$$

Example 5.3 8(a): Compute $\iint_S \vec{F} \cdot \vec{n} \, dA$ for $\vec{F} = xz\vec{i} - xy\vec{k}$, $S =$ graph of $z = xy$ $0 \le x \le 1, 0 \le y \le 2$

$\vec{G}(x,y) = (x, y, xy)$ $\quad G_x = (1,0,y)$ $\quad G_y = (0,1,x)$, $G_x \times G_y = (-y, -x, 1)$

$$\iint_S \vec{F} \cdot \vec{n} \, dA = \int_0^1 \int_0^2 (x^2 y, 0, -xy) \cdot (-y, -x, 1) \, dA$$

$$= \int_0^1 \int_0^2 (-x^2 y^2 - xy) \, dA$$

$$= -\int_0^2 \left(\frac{y^2}{3} + \frac{y}{2} \right) dA$$

$$= -\left(\frac{8}{9} + \frac{4}{4} \right) = -\frac{17}{9}$$