Lec 4/19

Wednesday, April 19, 2017 09:16

Differentiation of Fourier Series

$f:\mathbb{R}\to\mathbb{C}$ periodic & continuous (on all of $\mathbb{R}$) and piecewise C^1,

then the fourier coeffs of f' are $c_n' = inc_n$

if f is piecewise C^2 and continuous, then

$$\left(\sum_{n=-\infty}^{\infty} c_n e^{in\theta}\right)' = \sum_{n=-\infty}^{\infty} inc_n e^{in\theta}$$

$$\left(\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\theta) + \sum_{n=1}^{\infty} b_n \sin(n\theta)\right)' = \sum_{n=1}^{\infty} -na_n \sin(n\theta) + \sum_{n=1}^{\infty} nb_n \cos(n\theta)$$

$\Rightarrow$ Fourier series of f converges uniformly & absolutely:

note $|e^{in\theta}| = 1$

$$\left|\sum_{n=-N}^{N} c_n e^{in\theta}\right| \le \sum_{n=-N}^{N} |c_n| = \sum_{\substack{n=-N\\ n\neq 0}}^{N} \left|\frac{c_n'}{in}\right| = \sum_{\substack{n=-N\\ n\neq 0}}^{N} \frac{|c_n'|}{n}$$

now $\forall\, \alpha, \beta \in \mathbb{R}$, $\quad \alpha^2 - 2\alpha\beta + \beta^2 = (\alpha-\beta)^2 \ge 0$

$$\Rightarrow \frac{1}{2}(\alpha^2+\beta^2) \ge \alpha\beta$$

now take $\alpha = \frac{1}{n}$ $\quad \beta = |c_n'|$

$$\le \sum_{\substack{n=-N\\ n\neq 0}}^{N} \frac{1}{2}\left(\frac{1}{n^2} + |c_n'|^2\right) \Rightarrow \text{Fourier series for } f \text{ converges unf. \& abs.}$$

converges (p-test) converges (bessel's)

Note: periodic extension of $f(\theta) = \frac{\theta}{2}$ is not continuous

and $f(\theta) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(n\theta)$ does not converge absolutely or uniformly.

$\sum^{\infty} (-1)^{2k-1} \ldots k$

$f(\theta) = \sum_{n=1} \frac{\quad}{n} \sin(n\theta)$ does not converge ... uniformly.

Ex: $f(\frac{\pi}{2}) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\frac{\pi}{2}) = \sum_{k=1}^{\infty} \frac{(-1)^{2k-1}}{2k-1}(-1)^k$ does not converge absolutely.

If $c_0 \neq 0$ then $\int \sum_{n=-\infty}^{\infty} c_n e^{in\theta}\, d\theta$ can't be periodic

However if $c_0 = 0$ and $f: \mathbb{R} \to \mathbb{C}$ is piecewise smooth & periodic

Then $\int_0^{\theta} f(t)\,dt$ is 2π-periodic.

Modifications of Fourier series to functions of arbitrary period:

Suppose $g(x)$ is 2ℓ-periodic. Then $f(\theta) = g(\frac{\ell}{\pi}\theta)$ is 2π-periodic

$\left(f(\theta + 2\pi) = g(\frac{\ell}{\pi}\theta + 2\ell)\right)$

If g is piecewise smooth then

call this $x \Rightarrow \theta = \frac{\pi}{\ell}x$

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\theta) + \sum_{n=1}^{\infty} b_n \sin(n\theta) = g\left(\frac{\ell}{\pi}\theta\right)$$

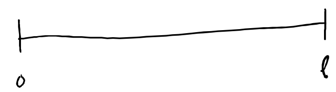
$$\Rightarrow g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{\ell}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{\ell}x\right)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta$$

$$= \frac{1}{\ell}\int_{-\ell}^{\ell} g(x)\cos\left(\frac{n\pi}{\ell}x\right)dx$$

Similarly $b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} g(x)\sin\left(\frac{n\pi}{\ell}x\right)dx$

Fourier studied the following problem.



Thin rod insulated with temperature varying along rod

$u(x,t)$ = temperature of rod at $x \in [0, \ell]$ at time $t \geq 0$

u satisfies the heat equation (1) $\dfrac{\partial u}{\partial t} = K \dfrac{\partial^2 u}{\partial x^2}$ for some $K > 0$.

(2) $\dfrac{\partial u}{\partial x}(0,t) = 0 = \dfrac{\partial u}{\partial x}(\ell, t)$ (heat does not leave ends of rod

(3) $u(x,0) = f(x)$ is initial temp. dist. along rod

(1) & (2) are additive conditions: if u_1, u_2 satisfy (1) and (2)

so does $\alpha u_1 + \beta u_2$. and any constant function satisfies (1) and (2)

Hence we expect that the solution of (1), (2), and (3) has to look like a constant + $u(x,t)$

$$\downarrow \text{ as } t \to \infty$$
$$0$$

i.e. rod will reach thermal equilibrium after a suff. long time.

Fourier looked for solutions of (1) and (2) that have the form

$$u(x,t) = \varphi(x)\psi(t)$$

$$\frac{\partial^2 u}{\partial x^2} = \varphi''(x)\psi(t)$$

$$\frac{\partial u}{\partial t} = \varphi(x)\psi'(t)$$

$$\Rightarrow \quad \varphi(x)\psi'(t) = K\varphi''(x)\psi(t)$$

$$\Rightarrow \quad \underset{\uparrow}{\frac{\psi'(t)}{K\psi(t)}} = \underset{\uparrow}{\frac{\varphi''(x)}{\varphi(x)}}$$

func of only t func of only x $\Rightarrow$ these two must be constant, and negative since $u(x, \infty-) = 0$.

$$\frac{\psi'(t)}{k\,\psi(t)} = -\lambda^2 = \frac{\varphi''(x)}{\varphi(x)}$$

$$\Rightarrow \quad \psi'(t) = -\lambda^2 k\,\psi(t) \quad \text{and} \quad \varphi''(x) = -\lambda^2 \varphi(x)$$

$$\Rightarrow \quad \psi(t) = C_0 e^{-\lambda^2 k t} \quad \text{and} \quad \varphi(x) = a\cos(\lambda x) + b\sin(\lambda x)$$

Plugging φ into (2), $\frac{\partial u}{\partial x} = \varphi'(x)\,\psi(t) \Rightarrow \varphi'(0) = \varphi'(\ell) = 0$

$\varphi'(x) = -a\lambda\sin(\lambda x) + b\lambda\cos(\lambda x)$, $\varphi'(0) = 0 \Rightarrow b = 0$

$\varphi(\ell) = 0 \Rightarrow \lambda \ell = n\pi \Rightarrow \lambda = \frac{n\pi}{\ell}$

upshot: $\varphi(x) = a_n \cos\left(\frac{n\pi}{\ell} x\right)$

Fourier guessed the general solution of (1), (2), and (3) would have to look like $u(x,t) = \sum_{n=0}^{\infty} a_n e^{-\frac{n^2\pi^2}{\ell^2} k t} \cos\left(\frac{n\pi}{\ell} x\right)$

(3) tells us $f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{\ell} x\right)$