Lec 4/17

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Theorem Let $f : \mathbb{R} \to \mathbb{C}$ be periodic & piecewise smooth:

(on $[-\pi, \pi]$ C' except at finitely many pts, and $f(\theta+), f(\theta-), f'(\theta+), f'(\theta-)$ exist $\forall \theta$).

Then the fourier series of the function always converges to $\frac{1}{2}[f(\theta+) + f(\theta-)]$.

Proof: Last time we showed that if $S_N^f(\theta) = \sum_{n=-N}^{N} c_n e^{in\theta}$

then $S_N^f(\theta) - \frac{1}{2}[f(\theta+) + f(\theta-)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} g_\theta(\varphi)\left(e^{(N+1)i\varphi} - e^{-Ni\varphi}\right) d\varphi = C'_{-N-1} - C'_N$

where C'_i are fourier coeffs of
$$g_\theta(\varphi) = \begin{cases} \dfrac{f(\theta+\varphi) - f(\theta-)}{e^{i\theta} - 1} & \varphi \in [-\pi, 0) \\ \dfrac{f(\theta+\varphi) - f(\theta+)}{e^{i\theta} - 1} & \varphi \in (0, \pi] \end{cases}$$

So if g_θ is integrable over $[-\pi, \pi]$

then $C'_{-N-1}, C'_N \to 0$ and the fourier series for f converges as claimed.

g_θ integrable $\Leftrightarrow$ $g_\theta(\varphi)$ is bounded (only need to check around $\varphi = 0$)

Follows: Use L'H: $\lim_{\varphi \to 0^-} g_\theta(\varphi) = \lim_{\varphi \to 0^-} \dfrac{f(\theta+\varphi) - f(\theta-)}{e^{i\theta} - 1} = \lim_{\varphi \to 0} \dfrac{f'(\theta+\varphi)}{ie^{i\varphi}} = \dfrac{f'(\theta-)}{i}$

Problem: L'H does not hold for complex-valued functions of a real variable.

Counterexample:

$$\lim_{t\to 0} \frac{t}{te^{i/t}} \stackrel{?}{=} \lim_{t \to 0} \frac{1}{e^{i/t} + t\left(\frac{-i}{t^2}\right)e^{i/t}} \cdot \frac{te^{i/t}}{te^{i/t}}$$

$$= \lim_{t \to 0} \frac{t}{t - i} e^{-i/t}$$

$$= \lim_{t \to 0} \frac{t(t+i)}{t^2 + 1}\left(\cos\left(\tfrac{1}{t}\right) - i\sin\left(\tfrac{1}{t}\right)\right)$$

$= 0$ by squeeze theorem.

On the other hand, $\lim_{t \to 0} \dfrac{t}{te^{i/t}} = \lim_{t \to 0} \dfrac{1}{e^{i/t}} = \lim_{t \to 0} \left(\cos\left(\tfrac{1}{t}\right) - i\sin\left(\tfrac{1}{t}\right)\right)$

Proposition (Left-hand Version): Suppose $f, g : [a,b] \to \mathbb{C}$ continuous, and smooth on (a,b). $f(b) = g(b) = 0$ and $f'(b-)$ exists, $g'(b-) \neq 0$, then

$$\lim_{t \to b} \frac{f(t)}{g(t)} = \frac{f'(b-)}{g'(b-)}$$

Proof: Apply MVT to real & imaginary parts of f & g.

If $t \in (a,b)$ then
$$\begin{aligned} f(t) = f(t) - f(b) &= \big(f_1(t) - f_1(b)\big) + i\big(f_2(t) - f_2(b)\big) \\ &= f_1'(\alpha_1)(t-b) + i f_2'(\alpha_2)(t-b) \\ &= \big(f_1'(\alpha_1) + i f_2(\alpha_2)\big)(t-b) \quad \text{where } \alpha_1, \alpha_2 \in (t,b) \end{aligned}$$

similarly $g(t) = \big(g_1'(\beta_1) + i g_2'(\beta_2)\big)(t-b)$ where $\beta_1, \beta_2 \in (t,b)$

So $$\lim_{t \to b^-} \frac{f(t)}{g(t)} = \lim_{t \to b} \frac{f_1'(\alpha_1) + i f_2(\alpha_2)}{g_1'(\beta_1) + i g_2'(\beta_2)} = \frac{f'(b-)}{g'(b-)}$$ since $t \to b^-$ $\Rightarrow \alpha_i \beta_i \to b^-$ $\square$

So applying the proposition we get desired result. $\square$

The proof required for convergence of fourier series that f is piecewise C^1 and $f(\theta+)$, $f(\theta-)$, $f'(\theta+)$, $f'(\theta-)$ exist.

Example: A periodic everywhere continuous function whose fourier series does not converge everywhere:

$$f(\theta) = \sum_{j=1}^{\infty} \frac{1}{j^2} \sin\left((2^{j^3} + 1) \frac{|x|}{2}\right)$$

This is continuous everywhere by the weierstrass M-test:

$$\left| \sum_{j=J}^{\infty} \frac{1}{j^2} \sin\left((2^{j^3}+1) \frac{|x|}{2}\right) \right| \leq \sum_{j=J}^{\infty} \frac{1}{j^2} \to 0 \text{ as } J \to \infty$$

however the fourier series of the function diverges at 0.

M-test: if $M_n > 0$, $\sum M_n < \infty$, and $|f_n(x)| \leq M_n$ then $\sum f_n(x)$ converges abs. & unif

Theorem (Carleson) If $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta$ is defined then Fourier series of f converges almost everywhere.

In particular, continuous functions

On the other hand, $\forall$ set of measure zero, can find a cts function whose Fourier series diverges on that set.

Remark: if f piecewise cts & the fourier series converges at θ, then it must converge to $\frac{1}{2}[f(\theta+) + f(\theta-)]$.