Lec 4/14

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Convergence of Fourier Series

f is 2π periodic

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} = \lim_{N\to\infty} \underbrace{\sum_{n=-N}^{N} c_n e^{in\theta}}_{S_N^f(\theta)}$$

If there is a reasonable interpretation then

$$c_n = \frac{1}{2\pi} \underbrace{\int_{-\pi}^{\pi} f(\theta) e^{-in\theta}}\, d\theta$$

this is defined for any integrable function.

__Theorem__ (Bessel's Inequality). If f is integrable & periodic then

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2\, d\theta$$

__Lemma__:

a) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, \overline{S_N^f(\theta)}\, d\theta = \sum_{n=-N}^{N} |c_n|^2$$

b) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(\theta)}\, S_N^f(\theta)\, d\theta = \sum_{n=-N}^{N} |c_n|^2$$

b) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} S_N^f(\theta)\, \overline{S_N^f(\theta)}\, d\theta = \sum_{n=-N}^{N} |c_n|^2$$

__Proof__

a) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) \sum_{n=-N}^{N} \overline{c_n}\, e^{-in\theta}\, d\theta = \sum_{n=-N}^{N} \overline{c_n}\, \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, e^{-in\theta}\, d\theta$$

$$= \sum_{n=-N}^{N} \bar{c}_n c_n$$

$$= \sum_{n=-N}^{N} |c_n|^2$$

b) $\frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(\theta)}\, S_N^f(\theta)\, d\theta = \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\, \overline{S_N^f(\theta)}} = \overline{\sum_{n=-N}^{N} |c_n|^2} = \sum_{n=-N}^{N} |c_n|^2$

c) $\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=-N}^{N} c_n e^{in\theta}\right)\left(\sum_{n=-N}^{N} \bar{c}_n e^{-in\theta}\right) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{m,n=-N}^{N} c_m \bar{c}_n\, e^{i(m-n)\theta}\, d\theta$

$$= \sum_{m,n=-N}^{N} c_m \bar{c}_n \; \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)}\, d\theta$$

Side: $\int_{-\pi}^{\pi} e^{i(m-n)\theta}\, d\theta = \left.\frac{e^{i(m-n)\theta}}{i(m-n)}\right|_{-\pi}^{\pi} = 0$ if $m \neq n$

$$= \int_{-\pi}^{\pi} 1\, d\theta = 2\pi \text{ if } m = n$$

$$= \sum_{n=-N}^{N} c_n \bar{c}_n = \sum_{n=-N}^{N} |c_n|^2$$

Proof of Bessel's inequality:

$$0 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta) - S_N^f(\theta)|^2\, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(f(\theta) - S_N^f(\theta)\right)\left(\overline{f(\theta)} - \overline{S_N^f(\theta)}\right) d\theta$$

$$= \frac{1}{2\pi} \int f(\theta)\overline{f(\theta)} - f(\theta)\overline{S_N^f(\theta)} - \overline{f(\theta)}\, S_N^f(\theta) + S_N^f(\theta)\overline{S_N^f(\theta)}\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\bar{f}(\theta) - f(\theta)\overline{S_N^f(\theta)} - \overline{f(\theta)}\, S_N^f(\theta) + S_N^f(\theta)\overline{S_N^f(\theta)}\, d\theta$$

$$\Rightarrow \sum_{n=-N}^{N} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2\, d\theta$$

Definition: $f: \mathbb{R} \to \mathbb{C}$ piecewise smooth if

(a) on any interval there is a finite set s.t. f is C^1 on the complement of this set.

(b) for all θ, the following one-sided limits exist and $\neq \pm\infty$

① $f(\theta+) = \lim_{\psi \to \theta^+} f(\psi)$

② $f(\theta-) = \lim_{\psi \to \theta^-} f(\psi)$

$\underset{③}{f'(\theta+)}$ and $\underset{④}{f'(\theta-)}$

Theorem: If f is 2π-periodic and piecewise cts in this sense then the Fourier series converges everywhere to $\frac{1}{2}[f(\theta+) + f(\theta-)]$

[**Remark**: $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2\, d\theta$ converges $\Rightarrow$ Fourier series for f converges to f almost everywhere.

Proof Strategy: Show that

$$\left| S_N^f(\theta) - \tfrac{1}{2}[f(\theta+) + f(\theta-)] \right| = C_N - C_{-N-1} \longrightarrow 0$$

by bessel-s inequality.

fourier coefficients of some periodic integrable function g.

Proof: n N π

function g.

Proof: $$S_N^f(\theta) = \sum_{n=-N}^{N} c_n e^{in\theta} d\theta = \sum_{m=-N}^{N} c_{-m} e^{-im\theta} \quad \text{and } c_{-m} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi) e^{im\psi} d\psi$$

$$= \sum_{m=-N}^{N} e^{-im\theta} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi)\, e^{im\psi} d\psi$$

$$= \int_{-\pi}^{\pi} \frac{1}{2\pi} \sum_{m=-N}^{N} f(\psi)\, e^{im(\psi-\theta)} d\psi \qquad \text{let } \varphi = \psi - \theta$$

$$= \int_{-\pi-\theta}^{\pi-\theta} f(\varphi+\theta)\left(\frac{1}{2\pi}\sum_{m=-N}^{N} e^{im\varphi}\right) d\varphi$$

Lemma (Exercise) if g is p-periodic & integrable

$$\int_a^{a+p} g = \int_b^{b+p} g$$

$$= \int_{-\pi}^{\pi} f(\varphi+\theta)\left(\frac{1}{2\pi}\sum_{m=-N}^{N} e^{im\varphi}\right) d\varphi$$

so $$S_N^f = \int_{-\pi}^{\pi} f(\varphi+\theta)\, D_N(\varphi)\, d\varphi$$ where $$D_N(\varphi) = \frac{1}{2\pi}\sum_{m=-N}^{N} e^{im\varphi}$$

Lemma

(a) $$\int_0^{\pi} D_N(\varphi)\, d\varphi = \int_{-\pi}^{0} D_N(\varphi)\, d\varphi = \frac{1}{2}$$

(b) $$D_N(\varphi) = \frac{1}{2\pi}\, \frac{e^{i(n+1)\varphi} - e^{-in\varphi}}{e^{i\varphi} - 1} \qquad \varphi \neq 0$$

Proof:

(a) $$\int_0^{\pi} D_N(\varphi)\, d\varphi = \frac{1}{2\pi}\sum_{m=-N}^{N} \underbrace{\int_0^{\pi} e^{in\varphi}\, d\varphi}_{=}$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} \partial\varphi + \sum_{n=1}^{\infty} \underbrace{\int_0^{\pi} \overbrace{e^{in\theta} + e^{-in\theta}}^{= \cos n\theta} \partial\theta}_{= \pi}\right]$$

$$= \frac{1}{2}$$

(b) In any field:

$$a + ar + ar^2 + \cdots + ar^k$$

$$= \frac{ar^{k+1} - ar}{r-1}$$

$$\sum_{m=-N}^{N} e^{im\theta} = e^{-iN\theta} + e^{-iN\theta} e^{i\theta} + \cdots + e^{-in\theta}(e^{i\theta})^{2N}$$

$$= \frac{e^{-iN\theta}(e^{i(2N+1)\theta}) - e^{-in\theta}}{e^{i\theta} - 1}$$

$$= \frac{e^{i(N+1)\theta} = e^{-in\theta}}{e^{i\theta} - 1}$$

Proof:

$$\frac{1}{2}\left[f(\theta+) + f(\theta-)\right] - S_N^{+}(\theta)$$

$$= \int_0^{\pi} f(\theta+) D_N(\varphi)\partial\varphi + \int_{-\pi}^{0} f(\theta-) D_N(\varphi)\, d\varphi$$

$$- \int_0^{\pi} f(\varphi+\theta) D_N(\varphi)\partial\varphi - \int_{-\pi}^{0} f(\varphi+\theta) D_N(\varphi)\, d\varphi$$

$$= \int_0^{\pi} \left(f(\theta+) - f(\varphi+\theta)\right) \frac{1}{2\pi} \frac{e^{i(N+1)\varphi} - e^{-iN\varphi}}{e^{i\varphi} - 1} \partial\varphi$$

$$= \int_0 \Big(\cdots$$

$$+ \int_{-\pi}^{0} \Big(f(\theta -) - f(\varphi + \theta) \Big) \frac{1}{2\pi} \frac{e^{i(N+1)\varphi} - e^{-iN\varphi}}{e^{i\varphi} - 1} \, d\varphi$$

Let $g(\varphi) = \dfrac{f(\theta +) - f(\varphi + \theta)}{e^{i\varphi} - 1}$

$\Rightarrow$ get $\underset{\text{of } g}{C_N} - \underset{\text{of } g}{C_{-N-1}}$