Lec 4/13

Thursday, April 13, 2017 09:12

$f : \mathbb{R} \to \mathbb{R}$ 2π - periodic

want to express

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

If there is a good fourier series,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} \, d\theta$$

If f is real valued, we'd like

$$f(\theta) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta))$$

well $c_n e^{in\theta} = c_n(\cos(n\theta) + i\sin(n\theta))$ n positive integer.

$c_{-n} e^{-in\theta} = c_{-n}(\cos(n\theta) - i\sin(n\theta))$

first, w/o c_n or c_{-n} we get

$$\sin(n\theta) = \frac{e^{in\theta} - e^{-in\theta}}{2i}$$

$$\cos(n\theta) = \frac{e^{in\theta} + e^{-in\theta}}{2}$$

guess

$$\Rightarrow \boxed{\begin{aligned} a_n &= c_n + c_{-n} \\ b_n &= i(c_n - c_{-n}) \end{aligned}}$$

So

$$f(\theta) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \frac{a_n}{2} e^{in\theta} + \sum_{n=1}^{\infty} \frac{a_n}{2} e^{-in\theta} + \sum_{n=1}^{\infty} \frac{b_n}{2i} e^{in\theta} - \sum_{n=1}^{\infty} \frac{b_n}{2i} e^{in\theta}$$

and

$$\sum_{n=\infty}^{\infty} c_n e^{in\theta} = c_0 + \sum_{n=1}^{\infty} c_n e^{in\theta} + \underbrace{\sum_{n=-\infty}^{-1} c_n e^{in\theta}}_{\sum_{n=1}^{\infty} c_{-n} e^{-in\theta}}$$

comparing coeffs, $\frac{1}{2}(a_n - i b_n) = \frac{a_n}{2} + \frac{b_n}{2i} = c_n$

$$\frac{1}{2}(a_n + i b_n) = \frac{a_n}{2} - \frac{b_n}{2i} = c_{-n}$$

also $c_0 = \frac{1}{2} a_0$

solving for a_n and b_n,

$$a_n = c_n + c_{-n} \quad \text{for } n = 0, 1, \ldots$$

$$b_n = i c_n - i c_{-n} \quad \text{for } n = 1, 2, \ldots$$

so $a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} \, d\theta + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{in\theta} \, d\theta$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \frac{e^{in\theta} + e^{-in\theta}}{2} \, d\theta$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta) \, d\theta$$

Similarly $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta) \, d\theta$

Example $f(\theta) = |\theta|$ on $[-\pi, \pi]$ extended to a 2π-periodic function.

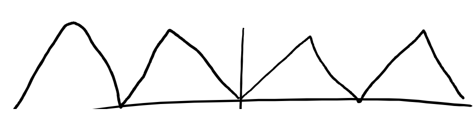
Remark: if f is odd $(f(-\theta) = -f(\theta))$

then $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(\theta) \cos(\theta)}_{\text{odd}} \, d\theta = 0$

if f is even $(f(-\theta) = f(\theta))$

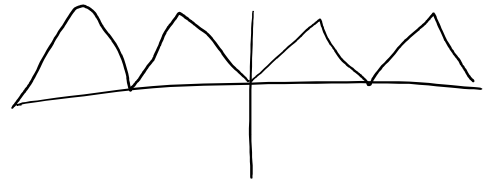
then $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{f(\theta) \sin(\theta)}_{\text{odd}} \, d\theta = 0$

$f(\theta) = |\theta|$ is even, and continuous everywhere,



So $b_n = 0$

everywhere,



so $b_n = 0$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi} \theta\cos(n\theta)\,d\theta$$

$$\overset{\text{IBP}}{=} \frac{2}{\pi}\left(\underbrace{\left[\theta\,\frac{\sin(n\theta)}{n}\right]_{\theta=0}^{\pi}}_{0} - \frac{1}{n}\int_0^{\pi}\sin(n\theta)\,d\theta\right)$$

$$= \frac{2}{n^2\pi}\left[\cos(n\theta)\right]_{n=0}^{\pi}$$

$$= \frac{2(-1)^n}{n^2\pi} - \frac{2}{n^2\pi}$$

so if n even, $a_n = 0$. if n odd, $a_n = \frac{-4}{n^2\pi}$

and $$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|\theta|\,d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi}\theta\,d\theta$$

$$= \frac{\pi}{2}$$

So $$f(\theta) \overset{?}{=} \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos((2n-1)\theta)}{(2n-1)^2}$$

this series converges absolutely since $|\text{each term}| \leq \frac{c}{n^2}$

$$\theta = 0 \Rightarrow 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\theta = 0 \Rightarrow \quad 0 = \frac{\quad}{2} - \frac{\quad}{\pi} \sum_{n=1} \frac{\quad}{(2n-1)^2}$$

$$\Rightarrow \quad \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi}{2}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Let $S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

$$\Rightarrow \quad S = \frac{\pi^2}{8} + \frac{1}{4} S$$

$$\Rightarrow \quad \frac{3}{4} S = \frac{\pi^2}{8}$$

$$\Rightarrow \quad S = \frac{\pi^2}{6}$$

<u>Def</u>: We say that $f : \mathbb{R} \to \mathbb{C}$ is piecewise continuous if over any finite interval, f has only finitely many discontinuities which are at most jump discontinuities. (i.e. both 1-sided limits exist). ///

<u>Bessel's Inequality</u>:

If f is 2π-periodic and piecewise continuous then

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 \, d\theta$$

$\Updownarrow$

converges

<u>Proof</u>:

$$\frac{1}{2\pi} \int^{\pi} \left| f(\theta) - \sum_{n=-N}^{N} c_n e^{in\theta} \right|^2 d\theta$$

$$\frac{}{2\pi}\int_{-\pi} \qquad \sum_{n=-N}$$

$$z\bar{z} = |z|^2$$