Lec 4/12

Wednesday, April 12, 2017 09:11

Relation of generalized[2] stoke's theorem to divergence theorem.

$S = \partial W$

If W is a region in $\mathbb{R}^n$ bounded by a $(k-1)$-dimensional hypersurface in $\mathbb{R}^n$.

Then Generalized stoke's theorem says if $\vec{\omega}$ is an $(k-1)$-form defined on $U \supseteq W$, then

$$\int_{\partial W} \vec{\omega} = \int_W d\vec{\omega}$$

$d\vec{\omega} = \left(\sum_{i=1}^n \frac{\partial}{\partial x_i} \vec{e}_i\right) \wedge \vec{\omega}$ is an n-form.

$\Lambda^n \mathbb{R}^n$ 1-dimensional w/ standard basis $\vec{e}_1 \wedge \vec{e}_2 \wedge \cdots \wedge \vec{e}_n$

an n-form is just $f(\vec{x})\ \vec{e}_1 \wedge \vec{e}_2 \wedge \cdots \wedge \vec{e}_n$

In particular, if $\vec{\omega}$ is an $(n-1)$-form, then $d\vec{\omega} = f(\vec{x})\ \vec{e}_1 \wedge \vec{e}_2 \wedge \cdots \wedge \vec{e}_n$

so $$\int_W d\vec{\omega} = \underbrace{\iint \cdots \int}_{n\text{-fold}} f(\vec{x})\, d^n\vec{x}$$

$\Lambda^{n-1} \mathbb{R}^n$ is n-dimensional w/ basis $\left\{ \vec{E}_i = \vec{e}_1 \wedge \cdots \wedge \vec{e}_{i-1} \wedge \vec{e}_{i+1} \wedge \cdots \wedge \vec{e}_n \right\}_{i=1}^n$ ($\vec{e}_i$ omitted.)

If $\vec{\omega}$ is an $(n-1)$-form $\sum_{i=1}^n g_i(\vec{x}) \vec{E}_i$, we define the dual vectorfied to be

$$\vec{\omega}^* = \sum_{i=1}^n (-1)^i g_i(x) \vec{e}_i.$$

If $n=3$, this is the correspondence we use to let $\times \cong \wedge$

$\vec{a}, \vec{b} \in \mathbb{R}^3$, $(\vec{a} \wedge \vec{b})^* = \vec{a} \times \vec{b}$

$\mathrm{div}(\omega^*) = d\omega$ n-form identified w/ scalar function.

If $\vec{G}: V \to \mathbb{R}^n$ ($V \subseteq \mathbb{R}^{n-1}$) parameterizes the hypersurface S

then $(\vec{G}_{u_1} \wedge \vec{G}_{u_2} \wedge \cdots \wedge \vec{G}_{u_{n-1}})^*$ is normal to the surface at every point.

and $\int \cdots \int |\vec{G}_{u_1} \wedge \vec{G}_{u_2} \wedge \cdots \wedge \vec{G}_{u_{n-1}}|\, du_1\, du_2 \cdots du_{n-1}$

and $\int \cdots \int_V |\vec{G}_{u_1} \wedge \vec{G}_{u_2} \wedge \cdots \wedge \vec{G}_{u_{n-1}}| \, du_1 \, du_2 \cdots du_{n-1}$

equal to the norm of the dual (both bases orthonormal)

is $(n-1)$-dim volume of S.

equal to $\omega^* \cdot (\vec{G}_{u_1} \wedge \cdots \wedge \vec{G}_{u_{n-1}})^*$

$$\int_S \vec{\omega} = \int \cdots \int_V \vec{\omega} \cdot \underbrace{(\vec{G}_{u_1} \wedge \vec{G}_{u_2} \wedge \cdots \wedge \vec{G}_{u_{n-1}}) \, du_1 \, du_2 \cdots du_{n-1}}_{\vec{n} \, dV_{n-1}}$$

and

$$\int_W d\vec{\omega} = \iint \cdots \int_W \operatorname{div}(\omega^*) \, d^n \vec{x}$$

Fourier Series

$f: \mathbb{R} \to \mathbb{C} = \mathbb{R}^2$ we say f is periodic w/ period P if

$f(x + nP) = f(x)$ $\forall x, n$. Can convert any to 2π-periodic functions

by $g(\theta) = f(\frac{P}{2\pi}\theta)$. Then $g(\theta + 2\pi) = f(\theta\frac{P}{2\pi} + P) = f(\theta\frac{P}{2\pi}) = g(\theta)$

So wolog only need to study 2π-periodic functions.

$n = 0, \pm 1, \pm 2, \ldots$

$e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$ basic examples of 2π-periodic functions.

Given a 2π-periodic function $f(\theta)$ is it true that

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \overset{def}{=} \lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{in\theta}$$

if f is real-valued we'd like to express this in the form

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\theta) + \sum_{n=1}^{\infty} b_n \sin(n\theta)$$

a_i, b_i real constants.

$\mathbb{R}^2$

a_i, b_i real constants.

first some discussion on differentiation & integration of $\mathbb{R} \to \mathbb{C} = \mathbb{R}^2$

if $f(\theta) = f_1(\theta) + i f_2(\theta)$ we define the derivative $f'(\theta) = f_1'(\theta) + i f_2'(\theta)$

and the indefinite integral $\int f(\theta)d\theta = \int f_1(\theta)d\theta + i\int f_2(\theta)d\theta$ + complex constant of integration

and the definite integral $\int_a^b \cdots$ . . . . . . —

<u>Lemma</u> (i) $(f(\theta)g(\theta))' = f'(\theta)g(\theta) + f(\theta)g'(\theta)$

(ii) $(cf(\theta))' = cf'(\theta)$

(iii) $\int cf(\theta)d\theta = c\int f(\theta)d\theta$

$\int_b^a \cdots \qquad \cdots$

(iv) $(e^{in\theta})' = in\, e^{in\theta}$

(v) $\int e^{in\theta}d\theta = \frac{1}{in}e^{in\theta}$ for $n \neq 0$

$\int_a^b \cdots \qquad \cdots$

<u>Proof</u>: easy. □

Assume that there is a Fourier series expansion $f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$

Assume we can integrate term by term.

$$f(\theta)\, e^{-ik\theta} = \sum_{n=-\infty}^{\infty} c_n e^{i(n-k)\theta}$$

$$\int_{-\pi}^{\pi} f(\theta)\, e^{-ik\theta} d\theta = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-k)\theta} d\theta$$

if $n \neq k$ $\int^{\pi} e^{i(n-k)\theta} d\theta = \frac{1}{i(n-k)} e^{i(n-k)\pi} - \frac{1}{i(n-k)} e^{i(n-k)(-\pi)} = 0$ by 2π-periodicity.

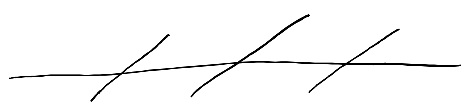
$$\text{if } n \neq k \quad \int_{-\pi}^{\pi} e^{i(n-k)\theta}\, d\theta = \frac{1}{i(n-k)} e^{i(n-k)\pi} - \frac{1}{i(n-k)} e^{i(n-k)(-\pi)} = 0 \quad \text{by } 2\pi\text{-periodicity.}$$

$$\text{if } n = k \quad \int_{-\pi}^{\pi} 1\, d\theta = 2\pi$$

$$\int_{-\pi}^{\pi} f(\theta)\, e^{-ik\theta}\, d\theta = c_k 2\pi$$

$$\Rightarrow \quad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, e^{-ik\theta}\, d\theta \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

Let $f(\theta) = \theta$ on $[-\pi, \pi)$ extended to $\mathbb{R}$ by 2π-periodicity.



$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta\, e^{-in\theta}\, d\theta$$

$$\overset{\text{IBP}}{=} \frac{1}{2\pi}\left(\left[\theta \frac{e^{-in\theta}}{-in} \right]_{-\pi}^{\pi} - \underbrace{\int_{-\pi}^{\pi} \frac{1}{-in} e^{-in\theta}\, d\theta}_{0} \right)$$

$$= \frac{1}{2\pi} \frac{2\pi}{-in} \qquad \text{for } n \neq 0$$

$$= \frac{i}{n}$$

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta\, d\theta = 0$$

So if f has a fourier series it is $f(\theta) = \sum_{\substack{n=-\infty \\ \neq 0}}^{\infty} \frac{i}{n} e^{in\theta}$