Lec 4/11

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Generalizations of Stoke's theorem for higher dimensions.

Area of surface $\vec{G} : U^{\subseteq \mathbb{R}^2} \to \mathbb{R}^n$ is $\iint_U \sqrt{|\vec{G}_u|^2 |\vec{G}_v|^2 - (\vec{G}_u \cdot \vec{G}_v)}\, du\, dv$

in $n=3$, this is $\iint_U |\vec{G}_u \times \vec{G}_v|\, du\, dv$ bc $|\vec{a} \times \vec{b}|^2 + (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 + (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$$

$$\sum_{1 \le i < j \le 3} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}^2 + (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$$

This holds in higher dimensions as well:

$$\underbrace{\sum_{1 \le i < j \le n} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}^2}_{} + (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$$

not the square of a norm of an n-dim vector.

However it is the square of the norm of an $\underset{\binom{n}{2}}{\frac{1}{2}n(n-1)}$-dim vector.

Replace cross product for $n=3$ by a higher dimensional analog.

$$\wedge : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{\binom{n}{2}}$$

reindex the standard basis of $\mathbb{R}^{\binom{n}{2}}$ as follows:

$\{\vec{e}_{ij}\}_{1 \le i < j \le n}$ ordered lexicographically.

e.g. $n=4$, $\binom{4}{2}=6$,

$\vec{e}_{12}$ $(1,0,0,0,0,0)$ $\quad$ $\vec{e}_{23} = (0,0,0,1,0,0)$

$\vec{e}_{13} = (0,1,0,0,0,0)$ $\quad$ $\vec{e}_{24} = (0,0,0,0,1,0)$

$\vec{e}_{14} = (0,0,1,0,0,0)$ $\quad$ $\vec{e}_{34} = (0,0,0,0,0,1)$

We'll denote $\mathbb{R}^{\binom{n}{2}}$ with this reindexing as $\Lambda^2\mathbb{R}^n$

Define the exterior product: $\wedge : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^{\binom{n}{2}}$

by $$\vec{a} \wedge \vec{b} = \sum_{1\le i<j\le n} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} e_{ij}$$

the identity becomes $|\vec{a} \wedge \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$

Properties of $\wedge$: $\vec{a} \wedge \vec{b} = -\vec{b} \wedge \vec{a}$

$\vec{a} \wedge \vec{a} = \vec{0}$

$(\alpha\vec{a} + \beta\vec{b}) \wedge \vec{c} = \alpha\vec{a} \wedge \vec{c} + \beta\vec{b} \wedge \vec{c}$

$\vec{e}_i \wedge \vec{e}_j = \vec{e}_{ij} = -\vec{e}_j \times \vec{e}_i$

$$|\vec{a} \wedge \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$$
$$= |\vec{a}|^2|\vec{b}|^2\sin^2\theta$$

So $|\vec{a} \wedge \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

Surface Area formula $\vec{G} : U \subseteq \mathbb{R}^2 \to \mathbb{R}^n$

$$A = \iint_U |\vec{G}_u \wedge \vec{G}_v| \, du \, dv$$

sign change to make this line up w/ $\times$

if $n=3$, $\wedge \cong \times$ where $e_1 = \vec{i}$ $e_2 = \vec{j}$ $e_3 = \vec{k}$ $e_1 \wedge e_2 = \vec{k}$ $e_1 \wedge e_3 = -\vec{j}$ $e_2 \wedge e_3 = \vec{k}$

Now Generalize surface integrals.

If $n=3$, then $\iint_S \vec{F} \cdot \vec{n}\, dA = \iint_S \vec{F} \cdot (\vec{G}_u \times \vec{G}_v)\, du\, dv$

to generalize, replace $\vec{G}_u \times \vec{G}_v$ by $\vec{G}_u \wedge \vec{G}_v$ $\binom{n}{2}$-dimensional.

Can't integrate n-dim v-field over a $\binom{n}{2}$-dim surface.

However, we can integrate an $\binom{n}{2}$-dim v.f.

Call a function $\vec{\omega} : U \xrightarrow{\subseteq \mathbb{R}^n} \Lambda^2 \mathbb{R}^n$ a 2-form on $\mathbb{R}^n$.

$$\vec{\omega}(\vec{x}) = \sum_{1 \le i < j \le n} \underset{\uparrow \text{ scalar}}{f_{ij}(\vec{x})}\, \vec{e}_i \wedge \vec{e}_j$$

If $S \subseteq \mathbb{R}^n$ is a 2-dim surface parametrized by $G : U \xrightarrow{\subseteq \mathbb{R}^2} \mathbb{R}^n$

also $\vec{\omega} : S \to \Lambda^2 \mathbb{R}^n$ is a 2-form then we define

$$\int_S \vec{\omega} = \iint_U (\vec{\omega} \circ \vec{G}) \cdot (\vec{G}_u \wedge \vec{G}_v)\, du\, dv$$

Now want to generalize Stoke's theorem.

$$\iint_S \underset{\downarrow \text{ analog of curl?}}{\operatorname{curl}(\vec{F})} \cdot \vec{n}\, dA = \underbrace{\int_{\partial S} \vec{F} \cdot d\vec{x}}_{\text{line integral, defined.}}$$

$\nabla \times \vec{F}$ in 3-dim, so $\nabla \wedge F$ in n-dim

$$\vec{F} \longrightarrow \left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} e_i \right) \wedge \vec{F} \qquad \vec{F} = \sum_{j=1}^{n} F_j e_j$$

$$\sum_{1 \le i < j \le n} \begin{vmatrix} \partial_{x_i} & \partial_{x_j} \\ F_i & F_j \end{vmatrix} e_i \wedge e_j =: d\vec{F}$$

$\uparrow$ exterior derivative of $\vec{F}$.

Stoke's theorem in higher dimensions:

$$\int_S d\vec{F} = \int_{\partial S} \vec{F} \cdot d\vec{x} \qquad \text{(subject to compatibility of orientations)}$$

Further generalizations:

Define $\Lambda^k \mathbb{R}^n$ k-th exterior power of $\mathbb{R}^n$, $k \leq n$

This is $\mathbb{R}^{\binom{n}{k}}$ with reindexed standard basis:

$$e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k} \qquad 1 \leq i_1 < i_2 < \dots < i_k \leq n$$

Have k-fold exterior product $(\mathbb{R}^n)^k \xrightarrow{\wedge} \Lambda^k \mathbb{R}^n$

define a k-form on $\mathbb{R}^n$ to be a function

$$\vec{w} : U^{\subseteq \mathbb{R}^n} \longrightarrow \Lambda^k \mathbb{R}^n$$

if $M^{\subseteq U}$ is a k-dimensional submanifold of $\mathbb{R}^n$ defined by $\vec{G} : W^{\subseteq \mathbb{R}^k} \longrightarrow \mathbb{R}^n$

Define $$\int_M \vec{w} = \underbrace{\iint \cdots \int_W}_{\text{k-fold integral}} (\vec{w} \circ \vec{G}) \cdot (\vec{G}_{u_1} \wedge \vec{G}_{u_2} \wedge \dots \wedge \vec{G}_{u_k}) \, du_1 \, du_2 \dots du_k$$

Generalized^2 Stoke's theorem:

$\vec{\sigma}$ is a k-1 form, $d\vec{\sigma} = \left(\sum_{i=1} \frac{\partial}{\partial x_i} e_i \right) \wedge \vec{\sigma}$

$$\int_M d\vec{\sigma} = \int_{\partial M} \vec{\sigma}$$