Lec 4/10

Monday, April 10, 2017 09:14

Integrating vector derivatives

Given $\vec{F} : \underset{\text{open}}{U} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ v.f., find potential.

Proposition: the following are equivalent:

(1) : $\int_C \vec{F} \cdot d\vec{x}$ depends only on endpoints of C

(2) : $\int_C \vec{F} \cdot d\vec{x} = 0$ if C is a closed curve

(3) : $\vec{F} = \nabla f$ for some $f : U \to \mathbb{R}$

Proof: (1) $\Rightarrow$ (2) if C closed, endpoints $= \vec{a}$, $\int_C \vec{F} \cdot d\vec{x} = \int_{\substack{\text{constant} \\ \text{curve at } \vec{a}}} \vec{F} \cdot dx = 0$

(2) $\Rightarrow$ (1) suppose C_1, C_2 both start at $\vec{a}$ and end at $\vec{b}$

then $C_1 \cup (-C_2)$ is a closed curve at $\vec{a}$

so $0 = \int_{C_1} \vec{F} \cdot d\vec{x} - \int_{C_2} \vec{F} \cdot d\vec{x}$

(3) $\Rightarrow$ (1) If $\vec{F} = \nabla f$ then $\int_C \vec{F} \cdot d\vec{x} = f(\vec{b}) - f(\vec{a}) = \text{Constant (ft } C\text{)}$.

(1) $\Rightarrow$ (3) Pick $\vec{a} \in U$, define $f : U \to \mathbb{R}$ by $f(\vec{x}) = \int_{C_{\vec{a},\vec{x}}} \vec{F} \cdot d\vec{x}$

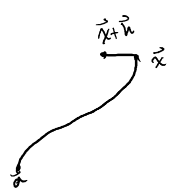
Where $C_{\vec{a},\vec{x}}$ is any curve btwn $\vec{a}, \vec{x}$.

Want to show $\nabla f = \vec{F}$.

Let $\vec{x} \in U$, $C_{\vec{a},\vec{x}}$ curve from $\vec{a}$ to $\vec{x}$:

Pick r small enough so that $\underset{\text{open}}{B}(r, \vec{x}) \subseteq U$

let $|h| < r$. Let $\vec{h} = (h, 0, \dots, 0)$. let $C_{\vec{x},\vec{x}+\vec{h}}$ be the



Straight line path from $\vec{x}$ to $\vec{x}+\vec{h}$

$\vec{g}(t) = (x_1 + t, x_2, \dots, x_n)$ where $0 \le t \le h$.

by (1), $f(\vec{x}+\vec{h}) = \int_{C \dots} \vec{F} \cdot d\vec{x}$

$$\text{by (1), } f(\vec{x}+\vec{h}) = \int_{C_{\vec{a},\vec{x}} \cup C_{\vec{x},\vec{x}+\vec{h}}} \vec{F}\cdot d\vec{x}$$

$$= \int_{C_{\vec{a}\vec{x}}} \vec{F}\cdot d\vec{x} + \int_{C_{\vec{x},\vec{x}+\vec{h}}} \vec{F}\cdot d\vec{x}$$

$$= f(\vec{x}) + \int_{C_{\vec{x},\vec{x}+\vec{h}}} \vec{F}\cdot d\vec{x}$$

$$\Rightarrow \frac{f(\vec{x}+\vec{h}) - f(\vec{x})}{h} = \frac{1}{h}\int_{C_{\vec{x},\vec{x}+\vec{h}}} \vec{F}\cdot d\vec{x}$$

$\downarrow$

$\partial_1 f(\vec{x})$ as $h \to 0$ $\quad = \frac{1}{h}\int_0^h \vec{F}(x_1+t,\dots,x_n)\cdot(1,0,\dots,0)\,dt$

$\|$

$H'(0)$ $\quad = \frac{1}{h}\int_0^h F_1(x_1+t,\dots,x_n)\,dt$

where

$H(h) = \int_0^h F_1(x_1+h,\dots,x_n)\,dt$ $\nearrow$

so $H'(h) = F_1(x_1+h,\dots,x_n)$

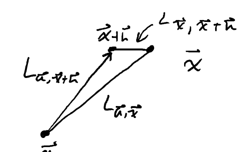
C^1 v.f. $\Rightarrow f$ is C^2

Remark: if $\vec{F} = \nabla f$ then $\partial_i F_j = \partial_j F_i$ by eq. of mixed partials. $(\partial_i\partial_j f = \partial_j\partial_i f)$
so $\partial_i F_j = \partial_j F_i \ \forall i,j$ is a necessary condition for $\vec{F} = \nabla f$.

Theorem: Suppose $U \subseteq \mathbb{R}^n$ is convex and $\vec{F}: U \to \mathbb{R}^n$ is a v.f. satisfying $\partial_i F_j = \partial_j F_i \ \forall i,j$ $(**)$
then $\vec{F} = \nabla f$.

Proof: Take $\vec{a} \in U$ and define $f(\vec{x}) = \int_{L_{\vec{a},\vec{x}}} \vec{F}\cdot d\vec{x}$ where $L_{\vec{a},\vec{x}}$ is straight line path from $\vec{x}$ to $\vec{a}$.

Want to show that $\nabla f(\vec{x}) = \vec{F}(\vec{x})$ for all $\vec{x} \in U$.



Choose $r > 0$ s.t. $B(r,\vec{x}) \subseteq U$. Let $0 < h < r$, $\vec{h} = (h,0,\dots,0)$.

If we show that $f(\vec{x}+\vec{h}) - f(\vec{x}) = \int_{L_{\vec{x},\vec{x}+h}} \vec{F}\cdot d\vec{x} = \int_0^h F_1(x_1+t,\dots,x_n)\,dt$ $\quad (*)$

then $\partial_1 f(\vec{x}) = F_1(\vec{x})$ follows as earlier.

$(*)$ is equivalent to $\int_{L_{\vec{a},\vec{x}} \cup L_{\vec{x},\vec{x}+h} \cup (-L_{\vec{a},\vec{x}+\vec{h}})} \vec{F}\cdot d\vec{x} = 0.$

This follows from green's theorem for $n=2$, stoke's theorem for $n=3$, and "generalized Stokes's theorem" for other n.

$n=3$: $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$ $\quad \operatorname{curl}(\vec{F}) = (\partial_2 F_3 - \partial_3 F_2)\vec{i} + (\partial_3 F_1 - \partial_1 F_3)\vec{j} + (\partial_1 F_2 - \partial_2 F_1)\vec{k} \overset{\text{by } (**)}{=} \vec{0}$.

so $\displaystyle\int_{L_{\vec{a},\vec{x}} \cup L_{\vec{x},\vec{x}+h} \cup (-L_{\vec{a},\vec{x}+\vec{h}})} \vec{F}\cdot d\vec{x} = \iint_S \vec{0}\cdot\vec{n}\, dA = 0$. we are done. $\square$

Generalized Version of Theorem: "convex" can be replaced by "simply connected".
i.e. any simple closed curve in U can be filled in with a disk.

If U not simply connected then $\partial_i F_j = \partial_j F_i$ is not sufficient for $\vec{F} = \nabla f$.