Lec 3/9

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Riemann Integral

Define $\iint_D f(x,y)\,dx\,dy$. $D \subseteq \mathbb{R}^2$ (nice enough.) f nice enough

Fubini Theorem → if D nice enough, f smooth enough.

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \int_c^d \chi_D(x,y) f(x,y)\,dy\,dx \quad \text{where } [a,b]\times[c,d] \supseteq D.$$

Change of variables:

stretch factor

$$\iint_S f(x,y)\,dx\,dy = \iint_T f(u,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$G(u,v) = (x(u,v), y(u,v)), \quad S = G(T)$$

if G a bijection, R nice enough, f smooth enough.

$\iint_E f(x,y)\,dx\,dy$ where E is solution set of $x^2+4xy+13y^2 \le 1$

$(x+2y)^2 + 9y^2 \le 1$

let $u = x+2y$

$v = 3y$

E is $u^2+v^2 \le 1$

$$\frac{\partial(u,v)}{\partial(x,y)}$$

$$(\text{jacobian matrix}) = \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3$$

but this is the wrong. need $\frac{\partial(x,y)}{\partial(u,v)}$.

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{3} \quad (\text{inverse mapping theorem}).$$

So $\iint_E f(x,y)\,dx\,dy = \iint_{Disk} f(u,v)\cdot \frac{1}{3}\,du\,dv = I$

take Disk to Rectangle. $u = r\cos\theta,\ v = r\sin\theta.$

So $I = \int_0^{2\pi}\int_0^1 f(r\cos\theta, r\sin\theta)\,\frac{r}{3}\,dr\,d\theta$ since r is jacobian.

$$\iint_E xy\,dx\,dy = \iint_D \left(u - \frac{2}{3}v\right)\left(\frac{1}{3}v\right)\left(\frac{1}{3}\,du\,dv\right)$$

$$= \frac{1}{9}\iint_D \left(uv - \frac{2}{3}v^2\right)du\,dv$$

$$= \frac{1}{9}\iint_R \left(r^2\cos\theta\sin\theta - \frac{2}{3}r^2\sin^2\theta\right)r\,dr\,d\theta$$

$$= \frac{1}{9}\int_0^{2\pi}\int_0^1 \left(r^3\cos\theta\sin\theta - \frac{2}{3}r^3\sin^2\theta\right)dr\,d\theta$$

$$= \frac{1}{9}\int_0^{2\pi} \frac{1}{4}\Big(\underbrace{\cos\theta\sin\theta}_{0} - \frac{2}{3}\underbrace{\sin^2\theta}_{\frac{1-\cos(2\theta)}{2}}\Big)\,dr\,d\theta$$

$\int_0^{2\pi} 1\,d\theta = 2\pi$

$$= \frac{1}{9}\cdot -\frac{1}{6}\cdot\frac{1}{2}\cdot 2\pi = -\frac{\pi}{54}$$

Derivatives. let $f(x)$ be cts on $[a,b]$. put $F(x) = \int_a^x f(t)\,dt$.

Theorem: $F'(x) = f(x)$

$G(x) = \int f(x,y)\,dy$

$F(x) = \int_0^1 \sin(xy)\,dy$

What is G'?

$$= -\frac{1}{x}\cos(xy)\Big|_0^1$$

$$= \frac{1-\cos(x)}{x}$$

hopefully

$$\frac{d}{dx} G(x) = \int_a^b \frac{\partial}{\partial x} f(x,y)\,dy .$$

$$\downarrow$$

$$F'(x)$$

$$\|$$

$$\int_0^1 \frac{\partial}{\partial x}\sin(xy)\,dy = \int_0^1 y\cos(xy)\,dy = \dots$$ ← hopefully, exercise.

Example in book where formula fails: $f(x,y) = \frac{x^2 y}{(x^2+y^2)^2}$ p. 188.

$(x,y) = (0,0)$ is a problem.

Maybe require f cts and maybe C' on region.

for each y, $x \mapsto f(x,y)$ is C' on interval.

$$G(x) = \int_{a(x)}^{b(x)} f(x,y)\,dy \qquad F(x) = \int_{x^2}^{x^3} \cos(xy)\,dy = \left(\frac{\sin(xy)}{x}\right)\Big|_{x^2}^{x^3}$$

$$= \frac{\sin(x^4) - \sin(x^3)}{x}$$

$$G'(x) = \frac{d}{dx}\int_{a(x)}^{b(x)} f(x,y)\,dy$$

$$= \underbrace{\int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,y)\,dy \cdot (b'(x) - a'(x))}_{\text{guess.}}$$

$$\boxed{\frac{d}{dt} F(x,y) = \frac{\partial F}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dt}}$$

Let $H(a,b,x) = \int_a^b f(x,y)\,dy$, $\quad \frac{\partial}{\partial x} H(x) = \frac{\partial H}{\partial x}\frac{dx}{dx} + \frac{\partial H}{\partial a}\frac{da}{dx} + \frac{\partial H}{\partial b}\frac{db}{dx}$

$$\frac{\partial H}{\partial x} = \int_a^b \frac{\partial}{\partial x} f(x,y)\,dy \qquad \frac{\partial H}{\partial a} = \frac{d}{da}\int_a^b f(x,y)\,dy = -f(x,a)$$

$$\frac{\partial}{\partial x} = \int_a \frac{\partial}{\partial x} f(x,y)\,dy \qquad \frac{\partial H}{\partial a} = \frac{\partial}{\partial a}\int_a f(x,y)\,dy = -f(x,a)$$

$$\frac{\partial H}{\partial b} = \frac{\partial}{\partial b}\int_a^b f(x,y)\,dy = f(x,b)$$

So $$\frac{\partial H}{\partial x} = \int_a^b \frac{\partial}{\partial x} f(x,y)\,dy + f(x,b)\frac{db}{dx} - f(x,a)\frac{da}{dx}$$

So $$G'(x) = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,y)\,dy + f(x,b(x))\,b'(x) - f(x,a(x))\,a'(x)$$

Classic Example: Dirichlet Integral:

$$\int \frac{\sin(x)}{x}\,dx$$ (no easy antiderivative).

but $$\int_0^\infty \frac{\sin(x)}{x}\,dx.$$ Converge?

$$\lim_{b\to\infty}\int_0^b \frac{\sin(x)}{x}\,dx$$ exist?

at $x=0$, $\lim_{x\to 0}\frac{\sin x}{x} = 1$. Integrate by parts:

$$\int \frac{\sin x}{x}\,dx \qquad u = \frac{1}{x} \qquad v = -\cos(x)$$
$$du = \frac{-1}{x^2} \qquad dv = \sin(x)$$

$$= \frac{-\cos(x)}{x} - \int \frac{\cos x}{x^2}\,dx$$

$$\int_a^b \frac{\sin x}{x}\,dx = \left.\frac{-\cos(x)}{x}\right|_a^b - \int_a^b \frac{\cos(x)}{x^2}\,dx$$

$$= \frac{-\cos(b)}{b} - \int_a^b \frac{\cos x}{x^2}\,dx + \text{constant}$$

since $\left|\int^b \frac{\cos(x)}{x^2}\right| \le \int^b \frac{1}{x^2} = \frac{1}{b} - \text{constant}$

$b \quad \int_a \frac{}{x^2} dx \quad$ constant

$\downarrow \qquad \downarrow \longrightarrow$ since $\left| \int_a^b \frac{\cos(x)}{x^2} \right| \le \int_a^b \frac{1}{x^2} = \frac{1}{b} - \text{constant}$

0 as $b \to \infty \qquad -\text{constant}$

So we know integral converges.

Harder ex: is this integral abs. convergent? no.

Clever:

$$F(t) = \int_0^\infty e^{-tx} \frac{\sin x}{x} \, dx$$

$$\frac{dF}{dt} = \int_0^\infty \frac{\partial}{\partial t}(e^{-tx}) \frac{\sin x}{x} \, dx$$

$$= \int_0^\infty -x \, e^{-tx} \frac{\sin x}{x} \, dx$$

$$= -\int_0^\infty e^{-tx} \sin x \, dx \qquad \text{Calc. 1} \quad \int e^{ax} \sin x \, dx \quad \text{IBP}$$

$$= \frac{e^{-tx}}{1+t^2} (t \sin(x) + \cos(x)) \Big|_{x=0}^{\infty}$$

$$= 0 - \frac{1}{1+t^2}(0+1) = \frac{-1}{1+t^2}$$

$$F'(t) = \frac{-1}{1+t^2} \Rightarrow F = -\arctan(x) + C.$$

Calculate C: $t = \infty$.

$$F(\infty) = 0 = -\underbrace{\arctan(\infty)}_{-\frac{\pi}{2}} + C \qquad \Rightarrow C = \frac{\pi}{2}$$

$$\int_0^\infty e^{-tx} \frac{\sin x}{x} dx = \frac{\pi}{2} - \arctan(t)$$

$$\int_0^\infty e^{-tx} \frac{\sin x}{x} \, dx = \frac{\pi}{2} - \arctan(t)$$

Set $t = 0$. $\quad F(0) = \int_0^\infty \frac{\sin(x)}{x} = \frac{\pi}{2}$.

needs justification.