Lec 3/31

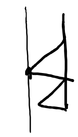
Friday, March 31, 2017 09:16

Surface Area :

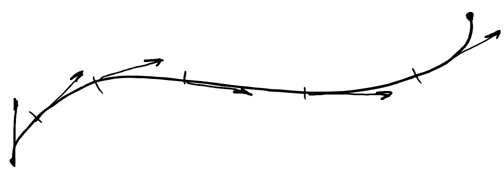
S surface in $\mathbb{R}^3$

$A(S) = \sup_P A_P(S)$? well if S = cylinder radius r, heignth

NO then $\sup_P A_P(S) = \infty$



Alternative way of approxing $L(C)$



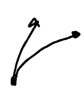
use sum of lengths of tangent lines

Let $\vec{g} : [a,b] \to \mathbb{R}^n$ be a C^1 parametric eqn for curve C.

take a partition $P = \{a = t_0 < \cdots < t_n = b\}$

Approximate $\vec{g}\big|_{[t_{i-1}, t_i]}$ by $\vec{g}_{tan}$ (eqn of tangent line at $t = t_{i-1}$.

$$\vec{g}_{tan}(t) = \vec{g}(t_{i-1}) + \vec{g}'(t_{i-1})\, t$$



length of segment from $\vec{g}_{tan}(t_{i-1})$ to $\vec{g}_{tan}(t_i)$ is $|\vec{g}'(t_{i-1})|(t_i - t_{i-1})$

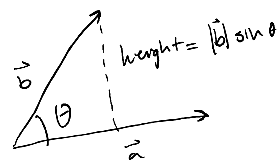
$$L(C) \approx \sum |\vec{g}(t_{i-1})| \overbrace{(t_i - t_{i-1})}^{\Delta t_i}$$

$$L(C) = \int_a^b |\vec{g}'(t)|\, dt$$

Analog for surfaces

Lemma If $\vec{a}, \vec{b} \in \mathbb{R}^3$ then $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$ = area of parallelogram 

Proof sketch

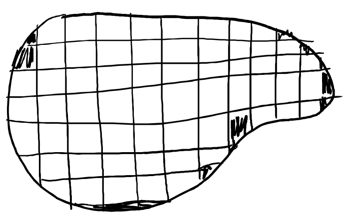


$\Rightarrow$ area of parallelogram $= |\vec{a}|\, h = |\vec{a}||\vec{b}| \sin\theta$

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 \qquad \text{Exercise}$$
$$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2 \cos^2\theta$$
$$= |\vec{a}|^2|\vec{b}|^2 \sin^2\theta$$

So, suppose our surface is given by a C^1 parametrized function $\vec{G} : \mathcal{R} \to \mathbb{R}^3$ ($\mathcal{R} \subseteq \mathbb{R}^2$ measurable)

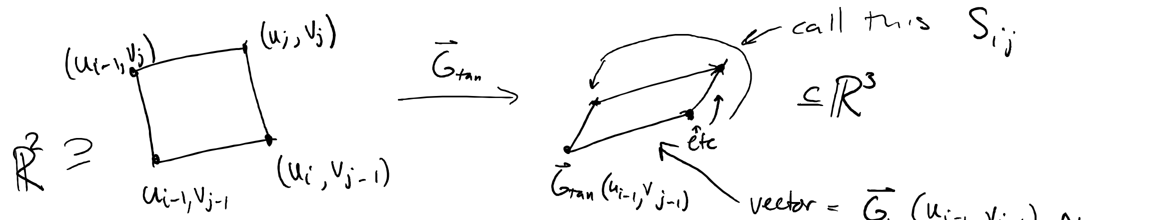
approximate $\mathcal{R}$ by rectangular grid (inner approx) w/ vertices (u_i, v_j)



approx $\vec{G}\big|_{[u_i - u_{i-1}] \times [v_j - v_{j-1}]}$ by $\vec{G}_{tan}$ where

$\vec{G}_{tan}$ is tangent plane approx to S at $\vec{G}(u_{i-1}, v_{j-1})$

$$\vec{G}_{tan}(u,v) = \vec{G}(u_{i-1}, v_{j-1}) + \vec{G}_u(u_{i-1}, v_{j-1})(u - u_{i-1}) + \vec{G}_v(u_{i-1}, v_{j-1})(v - v_{j-1})$$



so area of that, $A(S_{ij})$ is:

$$A(S_{ij}) \approx \left| \vec{G}_u(u_{i-1}, v_{j-1}) \times \vec{G}_v(u_{i-1}, v_{j-1}) \right| \Delta u_i \, \Delta v_j$$

So $$A(S) = \sum_{ij} A(S_{ij}) \approx \sum_{ij} \left| \vec{G}_u(u_{i-1}, v_{j-1}) \times \vec{G}_v(u_{i-1}, v_{j-1}) \right| \Delta u_i \, \Delta v_j$$

As $\Delta u_i, \Delta v_j \to 0$ these approxes become exact.

$$A(S) = \iint_{R} |\vec{G}_u \times \vec{G}_v| \, du\, dv$$

Check that this is correct for SA of cylinder rad r ht h:

$\vec{G}(u,v) = (r\cos u, r\sin u, v) \quad 0 \le u \le 2\pi, \quad 0 \le v \le h.$

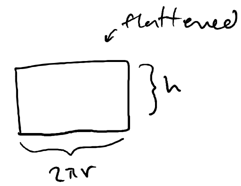
$\vec{G}_u = -r\sin u\,\vec{i} + r\cos u\,\vec{j} + 0\vec{k}$

$\vec{G}_v = 0\vec{i} + 0\vec{j} + \vec{k}$

$$\vec{G}_u \times \vec{G}_v = -r\sin u \underbrace{\vec{i}\times\vec{k}}_{=-\vec{j}} + r\cos u \underbrace{\vec{j}\times\vec{k}}_{=\vec{i}}$$

since perp, $\sin\theta = 1$ so $|\vec{G}_u \times \vec{G}_v| = r$

$$A = \int_0^{2\pi}\int_0^h r\,du\,dv = 2\pi r h$$



Remark: when we apply the formula for surface area to $\vec{G}: R \to \mathbb{R}^3$ we are assuming that $\vec{G}|_{R\setminus K}$ is one-to-one where K has content 0.

in above ex, $K =$ line $\{2\pi\} \times [0,h]$.

Special case: S is the graph of a C^1 function $f: R \to \mathbb{R}$

Can parametrize by $\vec{G}(x,y) = (x, y, f(x,y)) : R \to \mathbb{R}^3$

$\vec{G}_x = \vec{i} + f_x\vec{k} \qquad \vec{G}_y = \vec{j} + f_y\vec{k}$

$$\vec{G}_x \times \vec{G}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x\vec{i} - f_y\vec{j} + \vec{k}$$

So $|\vec{G}_x \times \vec{G}_y| = \sqrt{1 + f_x^2 + f_y^2}$.

$$A = \iint_R \sqrt{1+f_x^2+f_y^2}\, dx\,dy$$

Area of a sphere of radius R = 2× area of upper hemi:

$$f(x,y) = \sqrt{R^2-x^2-y^2}$$

$$S = 2\iint_{x^2+y^2\le R^2} \sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\; dx\,dy$$

$$= 2 \iint_{x^2+y^2\le R^2} \sqrt{\frac{R^2}{R^2-x^2-y^2}}\; dx\,dy$$

$$= 2 \int_0^{2\pi}\int_0^R \frac{Rr}{\sqrt{R^2-r^2}}\, dr\, d\theta$$

$$= R \int_0^{2\pi}\int_0^{R^2} \frac{1}{\sqrt{u}}\, du\, d\theta$$

$$= 2\pi R \; 2u^{1/2}\Big|_0^{R^2}$$

$$= 4\pi R^2$$