Lec 3/24

Friday, March 24, 2017 09:06

Application of Green's Theorem

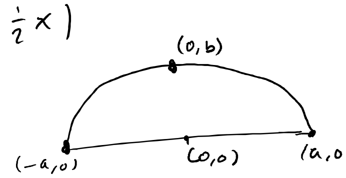
Computing area enclosed by a simple closed curve.

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial R} (P, Q) \cdot d\vec{g}.$$

If $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ then LHS is area (R)

$\int_a^b x\,dy$ $\quad$ $\int_a^b -y\,dx$ $\quad$ $\int_a^b \frac{1}{2}(x\,dy - y\,dx)$

Examples: $(P, Q) = (0, x), (-y, 0), (-\frac{1}{2}y, \frac{1}{2}x)$



Ex: Area enclosed by semi-ellipse

$$R = \{(x,y) : -a \le x \le a,\ 0 \le y \le b\sqrt{1 - \tfrac{x^2}{a^2}}\}$$

better parametrization of semi-ellipse:

C_1 top $\to$ $\vec{g}_1(t) = (a\cos(t), b\sin(t))$, $t \in [0, \pi]$.

C_2 bottom $\to$ $\vec{g}_2(t) = (t, 0)$ $\quad t \in [-a, a]$

take $(P, Q) = \frac{1}{2}(-y, x)$.

area is: $\frac{1}{2}\int_{C_1 \cup C_2} (-y\,dx + x\,dy)$

$C_1 : (dx, dy) = (-a\sin(t), b\cos(t))\,dt$

$-y\,dx + x\,dy = (ab\sin^2(t) + ab\cos^2(t))\,dt$

$= \frac{1}{2}\int_0^\pi ab\,dt$

$C_2 : (dx, dy) = (1, 0)$

$-y\,dx + x\,dy = 0$

$= \underline{\pi ab}$

$$= \frac{\pi a b}{2}$$

find centroid of semi-ellipse.

$$\bar{X}_i = \frac{\iint_R x_i \, dA}{A = \iint_R 1 \, dA}$$

$$A = \tfrac{1}{2}\pi a b$$

by symetry, $\bar{X} = 0$. need to find $\bar{y}$

$$\bar{y} = \frac{\iint_R y \, dA}{\frac{1}{2}\pi a b}$$

G.T. want $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y$. $(P, Q) = (0, xy)$.

$$= \frac{\oint_{\partial R} xy \, dy}{\frac{1}{2}\pi a b} = \frac{\int_{C_1} xy \, dy + \int_{C_2} xy \, dy}{\frac{1}{2}\pi a b}$$

$$= \left(\frac{1}{A}\right)\left(\int_0^{\pi} a\cos(t) b\sin(t) \, b\cos(t) \, dt + \int_{-a}^{a} 0 \, dt \right)$$

$$= \frac{1}{A} \int_0^{\pi} a b^2 \cos^2(t) \sin(t) \, dt$$

$$= \frac{1}{A} \int_{-1}^{1} a b^2 u^2 \, du$$

$$= \frac{1}{A} \, \frac{1}{3} a b^2 u^3 \Big|_{-1}^{1}$$

$$= \left(\frac{1}{2}\pi a b\right)^{-1} \frac{2}{3} a b^2$$

$$= \frac{4}{3\pi} b$$

Recall

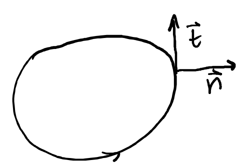
$$\int_C \vec F\cdot d\vec g = \int_a^b \underbrace{\vec F(\vec g(t))\cdot \underbrace{\frac{\vec g'(t)}{|\vec g'(t)|}}_{\text{unit tangent}}}_{F_{tang}} \overbrace{|\vec g'(t)|\,dt}^{ds} \qquad \vec F = (P, Q)$$

$$= (P,Q)\cdot\vec t \quad \text{unit tangent}$$

GT: $$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial R} F_{tang}\, ds$$

$$(P,Q)\cdot\vec n \quad \text{unit normal outwards}$$

Reformulation: $$\iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \int_{\partial R} F_{norm}\, ds$$



$\vec n = \vec t$ rotated 90° cw.

$\mathbb{R}^2 = \mathbb{C}$

multiply by $e^{i\theta}$ $\approx$ rotation of angle θ. ccw

multiplying by $-i$ $\approx$ cw rotation 90°.

so if $\vec t = (t_1, t_2) = \frac{\vec g'(t)}{|\vec g'(t)|}$ then $\vec n = (n_1, n_2) = (t_2, -t_1)$

$$(P,Q)\cdot(t_2, -t_1) = Pt_2 - Qt_1 = (-Q, P)\cdot(t_1, t_2) = (Q, P)\cdot\vec t = (-Q, P)\cdot\frac{\vec g'(t)}{|\vec g'(t)|}$$

$$\int_{\partial R}(P,Q)\cdot\vec n\, ds = \int_{\partial R}(-Q,P)\cdot d\vec g \overset{GT}{=} \iint_R \underbrace{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)}_{\text{divergence of } \vec F} dA$$

If $(P,Q) = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ then $\int_{\partial R}(P,Q)\cdot d\vec g = \iint_R (\partial_1\partial_2 f - \partial_2\partial_1 f)\, dA = 0$

$\Rightarrow \quad \int_C \nabla f \cdot d\vec{g} = 0$ if C is a simple closed curve.

More general result:

$$\int_C \nabla f \cdot d\vec{g} = f(\vec{g}(b)) - f(\vec{g}(a)) \qquad \text{(FTC for line integrals)}.$$

Proof: Chain rule + FTC.

$$\frac{\partial}{\partial t}\left(f(\vec{g}(t))\right) = \nabla f(\vec{g}(t)) \cdot \vec{g}'(t)$$

So $$\int_C \nabla f \cdot d\vec{g} = \int_a^b \nabla f(\vec{g}(t)) \cdot \vec{g}'(t)\, \partial t = \int_a^b \frac{\partial}{\partial t} f(\vec{g}(t))\, \partial t = f(\vec{g}(b)) - f(\vec{g}(a)). \quad \blacksquare$$