Lec 3/22

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$C \subseteq \mathbb{R}^n$ given by $\vec{g}([a,b])$ where $\vec{g}$ piecewise C'.

$f: S \to \mathbb{R}$ $\quad C \subseteq S \subseteq \mathbb{R}^n$

$$\int_C f\, ds \overset{\text{def}}{=} \int_a^b f(\vec{g}(t))\,|\vec{g}'(t)|\,dt$$

$$\int_C 1\, ds = L(C)$$ (does not depend on parametrization)

Definition a Reparametrization of C is $\lambda: [c,d] \to [a,b]$ 1-1, onto, C' and $\vec{h}(u) = \vec{g}(\lambda(u))$

Proposition $\int_C f\, ds$ is invariant under reparametrization.

i.e. $\int_c^d f(\vec{h}(u))\,|\vec{h}'(u)|\,du = \int_a^b f(\vec{g}(t))\,|\vec{g}'(t)|\,dt$

PROOF

(1) λ is order preserving (increasing). i.e $\vec{g}, \vec{h}$ specify same direction along curve.

$\vec{h}'(u) = \vec{g}'(\lambda(u))\,\lambda'(u)$ and λ' nonnegative

$|\vec{h}'(u)| = |\vec{g}'(\lambda(u))|\,\lambda'(u)$

$$\int_c^d f(\vec{h}(u))\,|\vec{h}'(u)|\,du = \int_c^d f(\vec{g}(\lambda(u)))\,|\vec{g}'(\lambda(u))|\,\lambda'(u)\,du$$

$$= \int_a^b f(\vec{g}(t))\,|\vec{g}'(t)|\,dt$$ where $t = \lambda(u)$, $dt = \lambda'(u)\,du$

(2) λ is order reversing (decreasing).

Same as above but λ' is nonpositive so we have $(-\lambda(u))$ instead. then the integral is just the negative of what it is otherwise, but the bounds are flipped so we negate it again, so its invariant. ∎

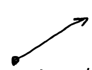
Line integrals over vector fields

$C \subseteq \mathbb{R}^n$ given by $\vec{g}([a,b])$ with $\vec{g}$ piecewise C'.

$F: U \to \mathbb{R}^n \qquad C \subseteq U \overset{\text{open}}{\subseteq} \mathbb{R}^n$

$$\int_C \vec{F} \cdot d\vec{g} \overset{\text{def}}{=} \int_a^b \vec{F}(\vec{g}(t)) \cdot \vec{g}'(t)\, dt \qquad \text{(line integral)}$$

$\vec{F}$: force field. i.e. gravity, electric field.



$\vec{x} \in U \subseteq \mathbb{R}^n \qquad \vec{F}(\vec{x}) = \text{force at } \vec{x}.$

Simplest case: motion along a straight line $\overrightarrow{\vec{a}\vec{b}}$.

Force is constant, in dir. of motion.

$W = |\vec{F}|\,|\vec{b} - \vec{a}|$ ← dist. moved.

↑ magnitude of force

Next simplest case: motion along a straight line $\overrightarrow{\vec{a}\vec{b}}$

Force is constant at angle θ to motion.

$W = (|\vec{F}|\cos(\theta))\,|\vec{b} - \vec{a}| = \vec{F} \cdot (\vec{b} - \vec{a})$

↑ component of force in dir. of motion.

General Case: motion along curve $C = \vec{g}([a,b])$

$\vec{F}: U \to \mathbb{R}^n$ arbitrary.

Partition $[a,b]$ into small subintervals

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$$W \approx \sum_{i=1}^{m} \vec{F}(\vec{g}(t_i)) \cdot (\vec{g}(t_i) - \vec{g}(t_{i-1}))$$

$$W = \int_C \vec{F} \cdot d\vec{g}$$

$$\vec{F} \cdot d\vec{g} = F_{\text{tangential}}\, ds \quad \text{where } F_{\text{tang}} = \vec{F} \cdot \frac{\vec{g}'(t)}{|\vec{g}'(t)|}$$

$$ds = |\vec{g}'(t)|\, dt$$

If C is reparametrized by $\lambda : [c,d] \to [a,b]$, $\vec{h}(u) = \vec{g}(\lambda(u))$

(1) if λ increasing then F_{tang} does not change

(2) If λ decreasing (reverses direction) then F_{tang} changes to $-F_{tang}$

so $\int_C \vec{F} \cdot d\vec{h} = -\int_C \vec{F} \cdot d\vec{g}$

Notation: $-C$ denotes C reparametrized by an order reversal.

Green's Theorem (in $\mathbb{R}^2$)

Jordan Curve theorem:

Definition We say that $g : [a,b] \to \mathbb{R}^n$ is a simple closed curve (continuous)
if g is 1-1 on (a,b) and $g(a) = g(b)$.

JCT: If $\vec{g} : [a,b] \to \mathbb{R}^2$ is a simple closed curve (C), then

$$\mathbb{R}^2 \setminus \vec{g}([a,b]) = \underbrace{U \sqcup V}_{\text{open \& connected}}$$

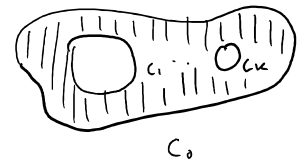
where one is bounded, the other unbounded.
↳ interior of C ↳ exterior of C

A region / subset (R) of $\mathbb{R}^2$ is enclosed by disjoint simple closed curves ($C_0, C_1, \dots C_k$)

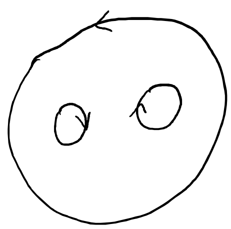
a region / subset of $\mathbb{R}^2$ is enclosed by disjoint simple closed curves,

if $C_1, C_2, \ldots, C_k \subseteq$ interior of C_0

R closed and $R \setminus \partial R = (\text{interior of } C_0) \cap \bigcap_{j=1}^{k} (\text{exterior of } C_j)$



Green's Theorem



Suppose $R \subseteq \mathbb{R}^2$ is closed & enclosed by a finite num of simple closed curves $C_0, \ldots, C_k$ which are piecewise C^1

and $\vec{F} = (P, Q) : U \to \mathbb{R}^2$ defined on $\overset{\text{open}}{U} \supseteq R$. Then

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \sum_{j=1}^{k} \int_{C_k} \vec{F} \cdot d\vec{s}_j$$

where C_0 is oriented CCW and $C_1, \ldots, C_k$ oriented CW.