Lec 3/21

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5.1 Integrals over curves in $\mathbb{R}^n$. (line integrals or Path integrals)

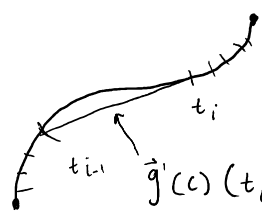
Integral for arc length.

C : $\vec{g}(t)$ $a \le t \le b$ parametrized curve in $\mathbb{R}^n$ (C^1 function)

<u>Definition</u> $L(C) = \int_a^b |\vec{g}'(t)| \, dt$

Physics Justification: $|\vec{g}'(t)|$ is speed of particle at time t.

Geometric derivation:



$P = \{t_1, \dots t_n\}$

$\vec{g}'(c)(t_i - t_{i-1})$ for some $c \in [t_i, t_{i-1}]$.

$$L_P(C) = \sum_{i=1}^{n} |\vec{g}(t_i) - \vec{g}(t_{i-1})|$$

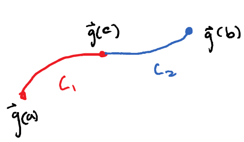
if $P \subseteq Q$ then $L_P(C) \le L_P(Q)$

define $L(C) = \sup_P L_P(C)$

We say C is rectifiable if $L(C)$ is finite.

If C is specified by $\vec{g} : [a,b] \to \mathbb{R}^n$ and $c \in [a,b]$

and C_1 is $\vec{g}|_{[a,c]}$ and C_2 is $\vec{g}|_{[c,b]}$ then $L(C) = L(C_1) + L(C_2)$.



If P is a partition of $[a,b]$, let $Q = P \cup \{c\}$.

$$Q_1 = Q \cap [a,c]$$
$$Q_2 = Q \cap [c,b]$$

Definition If $\vec{f} : [a,b] \longrightarrow \mathbb{R}^n$ we define

note: $(n=2,\ \mathbb{R}^2 = \mathbb{C})$

$$\int_a^b \vec{f}(t)\,dt = \left(\int_a^b f_1(t)\,dt,\ \int_a^b f_2(t)\,dt,\ \ldots,\ \int_a^b f_n(t)\,dt \right)$$

FTC: $$\int_a^b \vec{f}'(t)\,dt = \vec{f}(b) - \vec{f}(a)$$

Lemma $$\left| \int_a^b \vec{f}(t)\,dt \right| \le \int_a^b |\vec{f}'(t)|\,dt$$

Proof: Let $\vec{v} \in \mathbb{R}^n$
$$\left| \left(\int_a^b \vec{f}(t)\,dt \right) \cdot \vec{v} \right| = \left| \int_a^b \vec{f}(t) \cdot \vec{v}\,dt \right|$$
$$\le \int_a^b |\vec{f}(t) \cdot \vec{v}|\,dt \qquad \text{(Scaler version)}$$
$$\le \int_a^b |\vec{f}(t)||\vec{v}|\,dt \qquad \text{(Cauchy-Schwarz)}$$

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$= \left(\int_a^b |\vec{f}(t)|\,dt \right) |\vec{v}|$$

take $\vec{v} = \int_a^b \vec{f}(t)\,dt$

or: take $\vec{v}$ to be a unit vector.

So $$|\vec{v}|^2 \le \int_a^b |\vec{f}(t)|\,dt\ |\vec{v}|$$

$$\left| \int_a^b \vec{f}(t)\,dt \right| \le \int_a^b |\vec{f}(t)|\,dt$$

Theorem if $\vec{g} : [a,b] \longrightarrow \mathbb{R}^n$ is C', the curve $C = \vec{g}([a,b])$ is rectifiable.

and $$L(C) = \int_a^b |\vec{g}'(t)|\,dt$$

Proof: if P is a partition of $[a,b]$ then proof: if P is a partition of $[a,b]$, then

$$\begin{aligned} L_P(C) &= \sum_{i=1}^{n} |\vec{g}(t_{i-1}) - \vec{g}(t_i)| \\ &= \sum_{i=1}^{n} \left| \int_{t_{i-1}}^{t_i} \vec{g}'(t)\, dt \right| \\ &\le \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |\vec{g}'(t)|\, dt \\ &= \int_a^b |\vec{g}'(t)|\, dt \end{aligned} \quad \text{Part 1.}$$

So $\int_a^b |\vec{g}'(t)|\, dt$ is an upper bound $\xrightarrow{\sup_P L_P(C) \neq \infty}$ $\Rightarrow$ C is rectifiable.

If $[u,v] \subseteq [a,b]$ define $C_{u,v}$ to be $\vec{g}([u,v])$.

for $s \in [a,b]$, define $\varphi(s) = L(C_{a,s})$

Let $h>0$, small enough, and let $s \in (a,b)$.

$$\varphi(s+h) = L(C_{a,s+h}) = L(C_{a,s}) + L(C_{s,s+h}) = \varphi(s) + L(C_{s,s+h})$$

$$\begin{aligned} |\vec{g}(s+h) - \vec{g}(s)| = L_{\{s,s+h\}}(C_{s,s+h}) &\le \varphi(s+h) - \varphi(s) \\ &\le L(C_{s,s+h}) && \text{(from above} \\ &\le \int_s^{s+h} |\vec{g}'(t)|\, dt && \text{(part 1)} \\ &= |\vec{g}'(u)|\, h && \text{for some } u \in [s, s+h] \quad \text{(MVT 4 integrals)} \end{aligned}$$

So $\left| \dfrac{\vec{g}(s+h) - \vec{g}(s)}{h} \right| \le \dfrac{\varphi(s+h) - \varphi(s)}{h} \le |\vec{g}'(u)|$

$\lim\limits_{h\to 0^+} \left| \dfrac{\vec{g}(s+h) - \vec{g}(s)}{h} \right| = |\vec{g}'(s)|$, $\lim\limits_{h\to 0^+} |\vec{g}'(u)| = |\vec{g}'(s)|$.

So by sq. thm., $\lim\limits_{h\to 0^+} \dfrac{\varphi(s+h) - \varphi(s)}{h} = |\vec{g}'(s)|$

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$h \to 0^+ \qquad h$

(tweaks show that $\lim_{h\to 0^-} \dfrac{\varphi(s+h)-\varphi(s)}{h} = |\vec{g}'(s)|$)

So $\varphi'(s) = |\vec{g}'(s)|$, so by FTC,

$$L(C) = \varphi(b) - \varphi(a) = \int_a^b \varphi'(s)\,ds = \int_a^b |\vec{g}'(s)|\,ds. \qquad \blacksquare$$

More general Line Integrals:

Scalar line Integrals:

$C: \vec{g}:[a,b] \to \mathbb{R}^n$ parametrized curve, C^1 except for a finite # of corners.

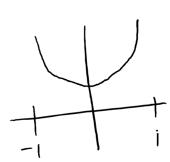
$f: S \to \mathbb{R}$, $C \subseteq S \subseteq \mathbb{R}^n$.

we define $\displaystyle\int_C f\,ds = \int_a^b f(\vec{g}(t))\,|\vec{g}'(t)|\,dt$

Centroid of a Curve C: $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$

$$\bar{x}_i = \frac{\int_C x_i\,ds}{\int_C ds} = \frac{\int_a^b g_i(t)\,|\vec{g}'(t)|\,dt}{\int_a^b |\vec{g}'(t)|\,dt}.$$

Example: find Centroid of $y = \cosh(x)$ $\quad -1 \le x \le 1$



$\bar{x} = 0$ by symmetry

$\vec{g}(t) = (t, \cosh(t))$

$\vec{g}'(t) = (1, \sinh(t))$

note: $\cosh^2(t) = \dfrac{1+\cosh(2t)}{2}$

$$|\vec{q}\,'(t)| = \sqrt{1 + \sinh^2(t)} = \cosh(t)$$

$$\bar{y} = \frac{\int_C y \, \partial s}{\int_C \partial s} = \frac{\int_{-1}^{1} \cosh^2(t)\, \partial t}{\int_{-1}^{1} \cosh(t)\, \partial t}$$

$$= \frac{\int_{-1}^{1} (1 + \cosh(2t))\, \partial t}{2\left(\sinh(1) - \sinh(-1)\right)}$$