Lec 3/2

Thursday, March 2, 2017 09:07

Theorem (Lebesgue): a bounded $f: [0,1] \to \mathbb{R}$ is R-integrable iff the set of points of discontinuity of f has measure zero.

It is easy to se that any finite set of intervals will not suffice to show $m(\mathbb{Q} \cap [0,1]) = 0$

(Adam)
Question: Is it true that if $S \subseteq [0,1]$ (nowhere dense) has measure zero then it has content zero.

Different Notions of Largeness:

1. Cardinality
2. "measure" — Exercise: Prove that $[0,1]$ does not have zero measure.
3. "topology"
4. "arithmetic" largeness
5. dimension(s)

Hausdorff dimension of Cantor set is $\frac{\log 2}{\log 3}$ $(m(C) = 0)$.

Exercise $C + C = [0,2]$, $C - C = [-1,1]$

$C + C = \{x + y : x, y \in C\}$, $C - C = \{x - y : x, y \in C\}$.

Two sets S_1, S_2 in $\mathbb{R}^n$ are homeomorphic if there is a 1-1 continuous (both ways) map $f: S_1 \to S_2$. (f is a homeomorphism). (Diffeomorphism if f is diffable).

$\mathbb{1}_C$ (indicator function). is it integrable? Yes.

but $\mathbb{1}_{\mathbb{Q}}$ not integrable (disc. everywhere)

Claim the set of points of discontinuity of 1_C has measure zero.

Claim Cantor set is closed. it's intersection of obviously closed sets.

1_C is cont. on $[0,1]\setminus C$, since $[0,1]\setminus C$ is open.

Exercise the set of points of discontinuity of C coincides w/ C

Exercise Generalized Cantor set is homeomorphic to C.

Exercise: what is total length of removed integrals for C. (1)

Exercise: $0 < \sum a_i = a < 1$ show this is not measure zero.

$\frac{a_0}{2}$ a_1 $\frac{a_2}{2}$

Integrals:

introduce polar coordinates.

$$\int_0^\infty e^{-x^2}\,dx \cdot \int_0^\infty e^{-y^2}\,dy = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dy\,dx$$

$$\stackrel{?}{=} \lim_{A\to\infty} \int_0^A\int_0^A e^{-(x^2+y^2)}\,dy\,dx$$ but what if A_1, A_2 differ?

Differentiation preserves "elementary"ness

Integration doesn't: $f(t) = \int_0^t e^{-x^2}\,dx$

$$= \int_0^t \frac{\sin x}{x}\,dx$$