

Lec 3/2

Thursday, March 2, 2017 09:07

Theorem (Lebesgue): a bounded $f: [0,1] \rightarrow \mathbb{R}$ is \mathbb{R} -integrable iff the set of points of discontinuity of f has measure zero.

It is easy to see that any finite set of intervals will not suffice to show $m(\mathbb{Q} \cap [0,1]) = 0$

Question (Adam): Is it true that if $S \subseteq [0,1]$ has measure zero then it has content zero.
 \nearrow nowhere dense

Different Notions of Largeness:

① Cardinality

② "measure"

③ "topology"

④ "arithmetic" largeness

⑤ dimension(s)

Exercise: Prove that $[0,1]$ does not have zero measure.

Hausdorff dimension of Cantor set is $\frac{\log 2}{\log 3}$ ($m(C) = 0$).

Exercise $C + C = [0,2]$, $C - C = [-1,1]$

$$C + C = \{x+y : x,y \in C\}, \quad C - C = \{x-y : x,y \in C\}.$$

Two sets S_1, S_2 in \mathbb{R}^n are homeomorphic if there is a 1-1 continuous (both ways) map $f: S_1 \rightarrow S_2$. (f is a homeomorphism). (Diffeomorphism if f is differentiable).

$\mathbb{1}_C$ (indicator function). is it integrable? Yes.

but $\mathbb{1}_a$ not integrable (disc. everywhere)

Claim the set of points of discontinuity of $\mathbb{1}_C$ has measure zero.

Claim Cantor set is closed. it's intersection of obviously closed sets.

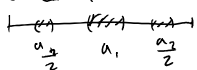
$\mathbb{1}_C$ is cont. on $[0,1] \setminus C$, since $[0,1] \setminus C$ is open.

Exercise The set of points of discontinuity of C coincides w/ C

Exercise Generalized Cantor set is homeomorphic to C .

Exercise: what is total length of removed intervals for C . (1)

Exercise: $0 < \sum a_i = a < 1$ show this is not measure zero.



Integrals:

$$\int_0^{\infty} e^{-x^2} dx \cdot \int_0^{\infty} e^{-y^2} dy = \iint_{0,0}^{\infty,\infty} e^{-(x^2+y^2)} dy dx$$

introduce polar coordinates.

$$\stackrel{?}{=} \lim_{A \rightarrow \infty} \int_0^A \int_0^A e^{-(x^2+y^2)} dy dx \quad \text{but what if } A_1, A_2 \text{ differ?}$$

Differentiation preserves "elementary"ness

Integration doesn't:

$$f(t) = \int_0^t e^{-x^2} dx = \int_0^t \frac{\sin x}{x} dx$$