Lec 3/10

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$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f + \int_c^b f \quad \text{where } f \text{ undefined at } c$$

improper

$$= \lim_{\varepsilon\to 0^-} \int_a^{c-\varepsilon} f$$

$$\int_{-1}^{1} \frac{1}{x}\,dx = 0 \quad \text{(odd function).}$$

principal value.

but $\int_0^1 \frac{1}{x}\,dx$ diverges.

$\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p > 1$. When $p=1$, $\int_1^x \frac{1}{t}\,dt = \log(x) \to \infty$.

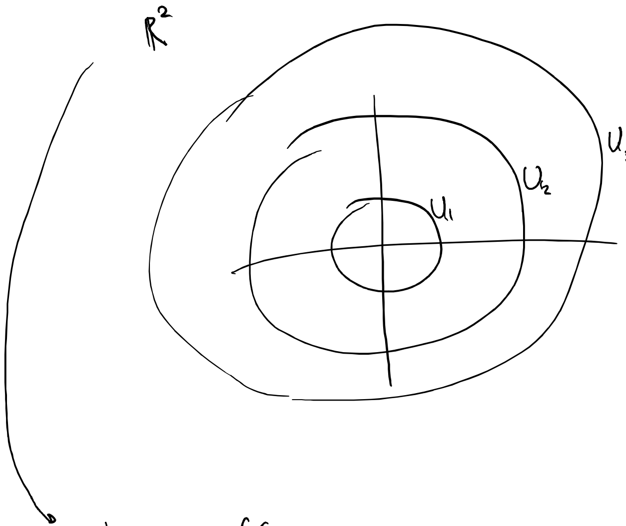
$$= \left.\frac{x^{-p+1}}{-p+1}\right|_1^\infty$$

. if $1-p > 0$ then diverges, if $1-p < 0$ then converges.

$$= \frac{1}{1-p} \text{ if converges.}$$

For line, only have 2 directions.

$$\iint f(x,y)\,dx\,dy$$



$$U_1 \subseteq U_2 \subseteq U_3 \subseteq \ldots$$ → unbounded in size.

$$\bigcup_{j=1}^{\infty} U_j = \mathbb{R}^2$$

Same as $\lim_{(a,b)\to(-\infty,\infty)}$ for $\mathbb{R}$.

$$\lim_{j\to\infty} \iint_{U_j} f \cdot dA$$

a different sequence of nested sets may yield a diff limit.

If $\underline{f(x,y) \geq 0}$ $\forall x,y$ and f integrable $\forall$ U_j and V_k sets, → unbounded sequences of measurable sets.

$$\lim_{j\to\infty} \iint_{U_j} f \cdot dA = \lim_{k\to\infty} \iint_{V_k} f \cdot dA = \iint_{\mathbb{R}^2} f \cdot dA.$$

Analogue is true for all $\mathbb{R}^n$, $n = 1, 2, \ldots$

General integrable $f(x)$, not necessarily ≥ 0.

$$\int_{-a}^{a} x\,dx = 0 \longrightarrow 0 \text{ as } a \to \infty. \quad \text{but} \quad \int_{a}^{\infty} x\,dx = \infty \longrightarrow \infty \text{ as } a \to \infty.$$

<u>Def</u>: $\int_S f\,dA$ is absolutely convergent if $\int_S |f|\,dA$ converges.

absolute $\Rightarrow$ regular convergence.

$f(x) = f^+(x) - f^-(x)$ where $f^+(x) = \begin{cases} f(x) & f(x) \geq 0 \\ 0 & \text{else} \end{cases}$ (similar for f^-).

$|f(x)| = f^+(x) + f^-(x)$

So use comparison test: $\int f^+$ and $\int f^-$ converge.

Given f, how to find formulas for f^+, f^-?

f measurable $\Rightarrow$ f^+ & f^-
↳ and $|f|$

$f^+(x) = \max(f(x), 0)$, $f^-(x) = \max(-f(x), 0)$

$f^+ = \dfrac{|f| + f}{2}$ $f^- = \dfrac{-|f| + f}{2}$ so yes.

f cts, g cts $\Rightarrow$ $m = \max(f,g)$ cts.

Famous Calculation:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = 1$$

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = \sqrt{2\pi}$$

$u = \frac{1}{\sqrt{2}}x$ $\sqrt{2}\,du = dx$

$$\sqrt{2}\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{2\pi}$$

$$\int_C e^{-z^2}\,dz$$

$z = x + iy$

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$$z^2 = x^2 + 2ixy - y^2$$

$$= \int_C e^{x^2} e^{2ixy} e^{-y^2} \quad ?$$

$\int e^{-x^2} \, dx$ not elementary

$\int \frac{\sin x}{x} \, dx$

$$J = \int_{-\infty}^{\infty} e^{-x^2} dx \quad , \quad J^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2 - y^2} dx \, dy$$

$$= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r \, dr \, d\theta$$

$$= 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^{\infty}$$

$$= \pi$$