Lec 2/9

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Paradox: Critical points / Hessian criteria for max/min work on open sets, but EVT only works on closed sets (compact sets).

Problem: Show that the rectangular box of fixed volume V and minimal surface area is a cube.

$V = xyz$ is constant, minimize $f(x,y,z) = 2xy + 2yz + 2zx$.

under constraint, this is $f(x,y) = 2xy + 2\frac{V}{y} + 2\frac{V}{x}$.

Minimizing over $S = \{(x,y) \mid x>0, y>0\}$ open quadrant, not compact.

$$0 = \frac{\partial f}{\partial x} = 2y - 2\frac{V}{x^2} \qquad \frac{\partial f}{\partial y} = 2x - 2\frac{V}{y^2} = 0$$

$$\hookrightarrow \quad y = \frac{V}{x^2}, \quad x = \frac{V}{y^2}, \quad y = \frac{V}{(V/y^2)^2} = \frac{y^4}{V} \Rightarrow Vy - y^4 = 0 \Rightarrow y = V^{1/3}$$

$$\Rightarrow x = \frac{V}{(V^{1/3})^2} = V^{1/3}, \quad z = V^{1/3}$$

So $(V^{1/3}, V^{1/3})$ is only crit. pt.

$$\frac{\partial^2 f}{\partial x^2} = 4\frac{V}{x^3} \qquad \frac{\partial^2 f}{\partial y^2} = 4\frac{V}{y^3} \qquad \frac{\partial^2 f}{\partial x \partial y} = 2$$

So $H\Big|_{cp} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$ so $\det H = 12 > 0$. There is a local max/min.

$\frac{\partial^2 f}{\partial x^2}\Big|_{cp} = 4 > 0$ so there is a local min at $(V^{1/3}, V^{1/3})$.

(eigenvalues are 2 & 6, both pos.)

but how to check for abs. min?

General method for finding absolute min on non-compact set.

(1) find local minima on S^{int} by finding critical points, analyzing Hessian.

(2) Assuming only finitely many crit pts, find the minimal local minimum. Call this $f(\vec{a})$ smallest local mini candidate for absolute min.

(3) find an bounded open set $U \subseteq S^{int}$ such that $\bar{U} \subseteq S$ and $a \in U$ so $\bar{U}$ is compact.

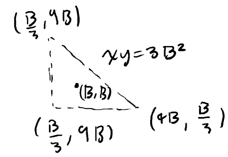
(4) Then show that if $\vec{x} \in S \setminus U$ then $f(\vec{x}) > f(\vec{a})$.

Now $f(\vec{a})$ is abs min on S.

Why: f has a min on the compact set $\bar{U}$, which can't be on the boundary since the boundary is in $S \setminus U$, and any point in the outside of $U > f(\vec{a})$.

back to Lec.

Let $B = \sqrt[3]{V}$. Local min at (B, B). $U =$



Notice that we can just minimize $\frac{1}{2}f$ in this situation. Call $\frac{1}{2}f$ f now.

If $(x,y) \in S \setminus U$ then one of the following conditions must hold:

1) $xy \geq 3B^2 \Rightarrow f(x,y) = xy + \frac{B^3}{x} + \frac{B^3}{y} > xy > 3B^2 = f(B,B)$.

2) $x \leq B/3 \Rightarrow f(x,y) = xy + \frac{B^3}{x} + \frac{B^3}{y} > \frac{B^3}{x} \geq 3B^2 = f(B,B)$

3) $y \leq B/3 \Rightarrow f(x,y) =$ " " " " " " " "

↳ something like this on take-home.

$f, g : U \xrightarrow{\subseteq \mathbb{R}^n} \mathbb{R}$.

Maxima & minima of multivariable function $f(\vec{x})$ subject to constraint $g(\vec{x}) = 0$

1) Solve for one var in terms of others in $g(\vec{x}) = 0$ (as in situation above), plug into $f(\vec{x})$. Now maxi minimizing an $n-1$ dimensional problem.

2) Lagrange multipliers:

Let $S = \{\vec{x} : g(\vec{x}) = 0\}$, f has a local extremum at $\vec{a} \in S$.

i.e. f has an extremum at $\vec{a}$ when restricted to $B(r, \vec{a}) \cap S$.

Suppose $\vec{v}$ is in tangent hyperplane to S at $\vec{a}$. then we can find a curve $\gamma : (-\varepsilon, \varepsilon) \to S$ so that $\gamma(0) = \vec{a}$ and $\gamma'(0) = \vec{v}$. Then $f \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}$ has a local extremum at 0. $\Rightarrow (f \circ \gamma)'(0) = 0$ so $\nabla f(\vec{a}) \cdot \vec{v} = 0$. Since $\vec{v}$ was arbitrary vector, $\nabla f(\vec{a})$ is perp. to the tangent hyperplane to S at $\vec{a}$. but $\nabla g(\vec{a})$ is also perpendicular to hyperplane. therefore, $\nabla f(\vec{a}) = \lambda \nabla g(\vec{a})$ for some real number λ.

This means $\vec{a}$ is a solution to the following system of eqns in n+1 variables:

1) $g(\vec{x}) = 0$

in $x_1, \dots, x_n, \lambda$.

2) $\nabla f(\vec{x}) - \lambda \nabla g(\vec{x}) = \vec{0}$

The Lagrangian function: $L_{f,g}(\vec{x}, \lambda) = f(\vec{x}) - \lambda g(\vec{x})$

$$0 = \frac{\partial L_{f,g}}{\partial \lambda} = 0 - g(\vec{x})$$

$$\vec{0} = \frac{\partial L_{f,g}}{\partial \vec{x}} = \nabla f(\vec{x}) - \lambda \nabla g(\vec{x})$$

Critical point of L.

<u>Problem</u>: Find the maximum & minimum of $f(x,y,z) = 2x^3 + 2y^2 + 3z^2$ on the sphere $x^2+y^2+z^2-1=0$ compact set.

$$L_{f,g}(x,y,z,\lambda) = 2x^3 + 2y^2 + 3z^2 - \lambda(x^2+y^2+z^2-1) = 0.$$

$$0 = \frac{\partial L}{\partial \lambda} = x^2+y^2+z^2-1$$

$$0 = \frac{\partial L}{\partial x} = 6x^2 - 2\lambda x = 2x(3x-\lambda)$$

$$0 = \frac{\partial L}{\partial y} = 4y - 2\lambda y = 2y(2-\lambda)$$

$$0 = \frac{\partial L}{\partial z} = 6z - 2\lambda z = 2z(3-\lambda)$$

$\Longrightarrow$