Lec 2/8

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Necessary condition for $f: U \to \mathbb{R}$ $\in C^2$ to have a local max/min at $\vec{a} \in U$:

$\vec{a}$ is a critical point; $\nabla f(\vec{a}) = 0$.

Not sufficient, use 2nd derivative test:

At a critical point $\vec{a}$:

$$f(\vec{a}+\vec{h}) = f(\vec{a}) + \underset{\substack{\text{linear} \\ \text{Terms of} \\ \text{Tay Pol}}}{\underset{\|}{0}} + \underbrace{\frac{1}{2}\sum_{|I|=2} \partial_I f(\vec{a})\vec{h}_I} + |\vec{h}|^2 \varepsilon_2(\vec{h})$$

Let $A = (\partial_I f(\vec{a}))$ the Hessian matrix (2nd partial derivatives)

$$= Q_A(\vec{h}) = \underset{1\times n}{\vec{h}^t} \underset{n\times n}{A} \underset{n\times 1}{\vec{h}}$$

Second derivative test: If $\vec{a}$ is a critical point then

1) if Q_A is positive definite then f has a local minimum at $\vec{a}$
   ($Q_A(\vec{h}) > 0 \ \forall \vec{h} \neq \vec{0}$)

2) " " negative definite " " local maximum "
   (" " < " ")

3) " indefinite " " saddle point "
   ($\exists \vec{h}_1, \vec{h}_2 \ Q_A(\vec{h}_1) > 0 > Q_A(\vec{h}_2)$)

4) " degenerate " The test is inconclusive
   ($\det(A) = 0$ and not indefinite)

Proof: Case 1: Let $Q_A(\vec{h})$ be positive definite.

if $\vec{h} \neq 0$, we can write $\vec{h} = |\vec{h}| \frac{\vec{h}}{|\vec{h}|} = |\vec{h}| \vec{u}$ ← unit vector

Then $Q_A(\vec{h}) = Q_A(|\vec{h}|\vec{u}) = |\vec{h}|^2 Q_A(\vec{u}) \geq m|\vec{h}|^2 > 0$

Note: $S^{n-1} = \{\text{unit vectors in } \mathbb{R}^n\}$ is compact $\Rightarrow$ Q_A has a minimum m.

$$f(\vec{a}+\vec{h}) = f(\vec{a}) + \frac{1}{2}|\vec{h}|^2 Q_A(\vec{u}) + |\vec{h}|^2 \varepsilon_2(\vec{h})$$
$$\geq f(\vec{a}) + \left(\frac{m}{2} + \varepsilon_2(\vec{h})\right)|\vec{h}|^2$$

On some open interval around $\vec{a}$, $\varepsilon_2(\vec{h}) > -\frac{m}{4}$.

$$\left.\begin{aligned} &\geq f(\vec{a}) + \frac{m}{4}|\vec{h}|^2 \\ &> f(\vec{a}) \end{aligned}\right\} \text{ on the open interval.}$$

So f has a minimum on the open interval, so a local min at $\vec{a}$. ■

When is $Q_A(\vec{h})$ positive/negative/in- definite or degenerate?

Spectral Theorem Let $Q_A : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form. Then there are n mutually perpendicular unit vectors $\vec{u}_1, \ldots, \vec{u}_n$ and real numbers $\lambda_1, \ldots, \lambda_n$ so that $\forall \vec{x} \in \mathbb{R}^n$:

$$Q_A(\vec{x}) = \sum_{i=1}^{n} \lambda_i (\vec{x}\cdot\vec{u}_i)^2$$

Moreover, λ_i are solutions to $p(\lambda) = \det(A - \lambda I_n) = 0$.

Note:
λ_i are eigenvalues
$\vec{u}_i$ are eigenvectors

Corollary:
1) Q_A is positive definite $\Leftrightarrow$ all $\lambda_i > 0$
2) Q_A is negative definite $\Leftrightarrow$ all $\lambda_i < 0$
3) Q_A is indefinite $\Leftrightarrow$ $\exists \lambda_i, \lambda_j$ s.t. $\lambda_i > 0 > \lambda_j$
4) Q_A is degenerate $\Leftrightarrow$ some $\lambda_i = 0$.

Proof Sketch for Spectral Theorem:

λ_1 = maximum value of Q_A on S^{n-1}

$\vec{u}_1$ = the vector where max value occurs.

Then take V_2 = vector subspace consisting of vectors perpendicular to $\vec{u}_1$.

λ_2 = max value of Q_A on $S^{n-1} \cap V_2$

$\vec{u}_2$ = where this occurs.

Continue until reach n.

Special case $n=2$.

$$A = \begin{pmatrix} \partial_1^2 f & \partial_1 \partial_2 f \\ \partial_2 \partial_1 f & \partial_2^2 f \end{pmatrix}$$ (evaluated at crit pt)

$$\det(A - \lambda I_2) = (\partial_1^2 f - \lambda)(\partial_2^2 f - \lambda) - (\partial_1 \partial_2 f)^2$$

$$= \lambda^2 - (\partial_1^2 f + \partial_2^2 f)\lambda + \partial_1^2 f \, \partial_2^2 f - (\partial_1 \partial_2 f)^2$$

$\det(A - \lambda I_2) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1 \lambda_2$

So comparing coeffs:

$$\lambda_1 + \lambda_2 = \partial_1^2 f + \partial_2^2 f$$

$$\lambda_1 \lambda_2 = \partial_1^2 f \, \partial_2^2 f - (\partial_1 \partial_2 f)^2$$

$$= \det(A)$$

Q_A indefinite $\iff \lambda_1 \lambda_2 < 0 \iff \det A < 0$

if $\det A = 0$ then A is degenerate

$\det A > 0 \Rightarrow \partial_1^2 f \partial_2^2 f > 0$ so they have the same sign.

Computer algebra algorithms for finding critical pts & eigenvalues.

If $f(\vec{x})$ is polynomial/rational, $\{\partial_i f(\vec{x}) = 0\}$ is a system of equations

$\overset{\text{Grobner}}{\Longrightarrow} \{q_j(\vec{x}) = 0\}$ simpler list of equations w/ same solutions

Example: 2.8 # 1 (g) $f(x,y,z) = x^3 - 3x - y^3 + 9y + z^2$

$0 = \partial_1 f = 3x^2 - 3 \Rightarrow x = \pm 1$ crit pts:

$0 = \partial_2 f = -3y^2 + 9 \Rightarrow y = \pm\sqrt{3}$ $\quad (1, \sqrt{3}, 0), (-1, \sqrt{3}, 0)$

$0 = \partial_3 f = 2z \Rightarrow z = 0$ $\quad (1, -\sqrt{3}, 0), (-1, -\sqrt{3}, 0)$

$\partial_1^2 f = 6x$ $\quad$ all mixed partials are 0.

$\partial_2^2 f = -6y$

$\partial_3^2 f = 2$ $\quad$ Hessian is $\begin{pmatrix} 6x & 0 & 0 \\ 0 & -6y & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad$ eigenvalues are $6x, -6y, 2$

Local min at $(1, -\sqrt{3}, 0)$

saddle point at all other points.