Lec 2/7

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Theorem: Let $\mathcal{U}$ be an open set in $\mathbb{R}^n$, $f: \mathcal{U} \to \mathbb{R}$ is of class C^{m+1} on $\mathcal{U}$

Then for any $\vec{a} \in \mathcal{U}$, $\displaystyle\lim_{\vec{h}\to\vec{0}} \frac{\left|f(\vec{a}+\vec{h}) - \sum_{|I|\le m} \frac{1}{|I|!}\partial_I f(\vec{a})\vec{h}_I\right|}{|\vec{h}|^m} = 0$

aka: $f(\vec{a}+\vec{h}) = \sum_{|I|\le m} \partial_I f(\vec{a})/|I|! \; \vec{h}_I + |\vec{h}|^m \varepsilon_m(\vec{h})$ where $\lim_{\vec{h}\to\vec{0}} \varepsilon_m(\vec{h}) = 0$.

Proof Strategy: reduce to single-var case.

Proof: Choose $r>0$ so $\overline{B(r,\vec{a})} \subseteq \mathcal{U}$. Choose $\vec{h} \in B(r,\vec{0}) \setminus \{\vec{0}\}$

$g_{\vec{a},\vec{h}}(t) = f(\vec{a}+t\vec{h})$, $t \in \left(-\frac{r}{|\vec{h}|}, \frac{r}{|\vec{h}|}\right)^{\ni 1}$

Lemma 2: for $0 \le k \le m+1$, $g^{(k)}_{\vec{a},\vec{h}}(t) = \sum_{|I|\le k} \partial_I f(\vec{a}+t\vec{h})\vec{h}_I$

Proof of Lemma 2: Induction on k.

$k=0$: $g(\vec{a},\vec{h})(t) = f(\vec{a}+t\vec{h})$ true by def.

induction: assume true for $k \le m$, show is true for $k+1$

$$g^{(k+1)}_{\vec{a},\vec{h}}(t) = \frac{d}{dt} g^{(k)}_{\vec{a},\vec{h}}(t) = \frac{d}{dt}\left(\sum_{|I|\le k} \partial_I f(\vec{a}+t\vec{h})\vec{h}_I\right)$$

$$= \sum_{|I|\le k} \frac{\partial}{\partial t}\left(\partial_I f(\vec{a}+t\vec{h})\right)\vec{h}_I$$

$$= \sum_{|I|\le k} \sum_{j=1}^{n} \partial_j \partial_I f(\vec{a}+t\vec{h}) \frac{\partial}{\partial t}(a_j + t h_j)\, \vec{h}_I$$

$$= \sum_{|I|\le k} \sum_{j=1}^{n} \partial_{j*I} f(\vec{a}+t\vec{h})\, \vec{h}_{j*I}$$

$$= \sum_{|I|\le k+1} \partial_I f(\vec{a}+t\vec{h})\, \vec{h}_I$$

So by induction it is true. ∎

back to theorem proof:

by single variable taylor theorem:

remainder term.

$$g_{\vec a,\vec h}(t) = \sum_{k=0}^{m} \frac{1}{k!} g_{\vec a,\vec h}^{(k)}(0)\, t^k + R(t)$$

$t=1$

$$g_{\vec a,\vec h}(1) = \sum_{k=0}^{m} \frac{1}{k!} \sum_{|I|=k} \partial_I f(\vec a+\vec h)\, \vec h_I + R(1)$$

$$= \sum_{|I|\le m} \frac{1}{|I|!} \partial_I f(\vec a+\vec h)\, \vec h_I + R(1)$$

$$\frac{\left| f(\vec a+\vec h) - \sum_{|I|\le m} \frac{1}{|I|!} \partial_I f(\vec a+\vec h)\, \vec h_I \right|}{|h|^m} = \frac{|R(1)|}{|\vec h|^m}$$

So it reduces to showing $\lim_{\vec h\to\vec 0} \frac{|R(1)|}{|\vec h|^m} = 0$

Lagrange form of $R(t)$:

$$R(t) = \frac{g_{\vec a,\vec h}^{(m+1)}(c)}{(m+1)!} t^{m+1} \quad \text{where } c \text{ btwn } 0 \text{ and } t.$$

$$R(1) = \frac{\sum_{|I|= m+1} \partial_I f(\vec a+c\vec h)\, \vec h_I}{(m+1)!} \quad \text{where } c \in (0,1).$$

$\vec a + c\vec h \in \overline{B(r,\vec a)}$ which is a compact set, so by EVT

$|\partial_I f(\vec a+c\vec h)| \le M_I$ which is maximal value bc of compactness.

$|\vec h_I| = |h_{i_1}||h_{i_2}| \cdots |h_{i_{m+1}}| \le |\vec h|^{m+1}$

So $R(1) \le M_I |\vec h|^{m+1}$

so $\lim_{\vec h\to\vec 0} \frac{|R(1)|}{|h|^m} \le \lim_{\vec h\to\vec 0} \frac{\sum M_I |h|^{m+1}}{(m+1)!\,|h|^m} = \lim_{\vec h\to\vec 0} \frac{\sum M_I}{(m+1)!} |h| = 0.$ ∎

Definition: $f: U \to \mathbb{R}$ ($U \subseteq \mathbb{R}^n$ open) has a local maximum or minimum at $\vec a \in U$ if f has a maximum or minimum at $\vec a$ when restricted to $B(r,\vec a)$ for some $r>0$.

Clearly: If f has a local max/min at $\vec a$ then f has a local max/min when restricted to a line in any coordinate direction. $\Rightarrow \partial_i f(\vec a) = 0$ for $i=1,\dots,n \Rightarrow \nabla f(\vec a) = 0$.

Definition $f: U \to \mathbb{R}$ has a critical point at $\vec a \in U$ if $\nabla f(\vec a)$

for $i=1,\dots,n \Rightarrow \nabla f(\vec{a})=0$.

Definition $f: U \to \mathbb{R}$ has a critical point at $\vec{a} \in U$ if $\nabla f(\vec{a}) = 0$.

This is a necessary but not sufficient condition for being a local max/min.

Additional complication: at a critical point, f can have local maximum in some directions & local minimum in other directions. In that case we say f has a saddle point at $\vec{a}$.

Example: $f(x,y) = x^2 - y^2$ a pringle 

To analyze whether we have a local max/min or saddle at crit pt, we need 2nd derivative test

at a critical point $\vec{a}$, the second order Taylor expansion looks like

$$f(\vec{a}+\vec{h}) = f(\vec{a}) + 0 + \frac{1}{2}\sum_{i,j=1}^{n} \partial_i \partial_j f(\vec{a})\, h_i h_j + |\vec{h}|^{2?} \xi_2(\vec{h}).$$

lower terms disappear

Definition let $A = (a_{ij})$ $n\times n$ symmetric matrix (i.e. $a_{ij} = a_{ji}$).

Then Quadratic form associated to A is:

$$Q_A(\vec{x}) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

We say that Q_A is positive definite if $Q_A(\vec{x}) > 0$ $\forall \vec{x} \neq 0$

" " negative definite " $Q_A(\vec{x}) < 0$ $\forall \vec{x} \neq 0$

indefinite if $Q_A(\vec{x}_1) > 0$, $Q_A(\vec{x}_2) > 0$

Degenerate if $\det A = 0$