Lec 2/6

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Implicit functions:

$$\left.\begin{aligned} F_1(x_1,\dots,x_n,y_1,\dots,y_m) &= 0 \\ F_2(x_1,\dots,x_n,y_1,\dots,y_m) &= 0 \\ &\vdots \\ F_m(x_1,\dots,x_n,y_1,\dots,y_m) &= 0 \end{aligned}\right\} \quad \vec{F}(\vec{x},\vec{y}) = \vec{0}.$$

Definition Suppose $U \subseteq \mathbb{R}^{m+n}$ open, $\vec{F}: U \to \mathbb{R}^m$ differentiable.
We say that $\vec{g}: \overset{\text{open}}{V} \subseteq \mathbb{R}^n \to \overset{\text{open}}{W} \subseteq \mathbb{R}^m$ is an implicit function defined by $\vec{F}(\vec{x},\vec{y}) = \vec{0}$ if:

1) $V \subseteq \mathbb{R}^n$, $W \subseteq \mathbb{R}^m$ open

2) $V \times W \subseteq U$

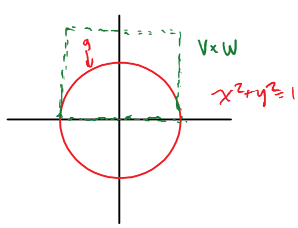
3) if $(\vec{x},\vec{y}) \in V \times W$ then $\vec{F}(\vec{x},\vec{y}) = \vec{0} \iff \vec{y} = \vec{g}(\vec{x})$

$\Updownarrow$

3) $\{(\vec{x},\vec{y}) \mid \vec{F}(\vec{x},\vec{y}) = \vec{0}\} \cap (V \cap W) = \{(\vec{x}, \vec{g}(\vec{x})) \mid \vec{x} \in V\}$

Example: $g(x) = \sqrt{1-x^2}$, $g: (-1,1) \to (0,2)$

implicit function defined by $x^2 + y^2 = 1$



Want to express Jacobian $D\vec{g}$ in terms of $D\vec{F}$.

Notation: $\vec{G}$ some diffable function defined on some open subset contained in $\mathbb{R}^q$ and taking values in $\mathbb{R}^p$

$\vec{v} = \vec{G}(\vec{u})$, $\quad D\vec{G}(\vec{u}) = \dfrac{\partial \vec{G}}{\partial u} = \dfrac{\partial(G_1,\dots,G_p)}{\partial(u_1,\dots,u_q)} = (\partial_j G_i)_{\substack{1\le i\le p \\ 1\le j\le q}}$

let $\vec{\lambda}: V \longrightarrow U$ be defined to be $\vec{\lambda}(\vec{x}) = (\vec{x}, \vec{g}(\vec{x}))$

$$(\vec{F} \circ \vec{\lambda})(\vec{x}) = \vec{F}(\vec{x}, \vec{g}(\vec{x})) = 0 \quad \text{for all } \vec{x} \in V.$$

$$\text{so } \underset{m \times n}{0} = D\vec{F}(\vec{x}, \vec{g}(\vec{x})) \cdot D(\vec{\lambda}(\vec{x})) = \begin{pmatrix} \frac{\partial \vec{F}}{\partial \vec{x}} & \frac{\partial \vec{F}}{\partial \vec{y}} \end{pmatrix} \begin{pmatrix} I_n \\ \frac{\partial \vec{g}}{\partial \vec{x}} \end{pmatrix} = \underset{m \times n}{\left(\frac{\partial \vec{F}}{\partial \vec{x}}\right)} + \underset{m \times m}{\left(\frac{\partial \vec{F}}{\partial \vec{y}}\right)} \underset{m \times n}{\left(\frac{\partial \vec{g}}{\partial \vec{x}}\right)}$$

$$\Rightarrow \left(\frac{\partial \vec{F}}{\partial \vec{y}}\right)\left(\frac{\partial \vec{g}}{\partial \vec{x}}\right) = \left(\frac{\partial \vec{F}}{\partial \vec{x}}\right)$$

these are evaluated at $(\vec{x}, \vec{g}(\vec{x}))$

$$\Rightarrow D\vec{g}(\vec{x}) = \left(\frac{\partial \vec{g}}{\partial \vec{x}}\right) = -\left(\frac{\partial \vec{F}}{\partial \vec{y}}\right)^{-1} \cdot \left(\frac{\partial \vec{F}}{\partial \vec{x}}\right)$$

to do this, $\det\left(\frac{\partial \vec{F}}{\partial \vec{y}}(\vec{x}, \vec{g}(\vec{x}))\right)$ must $\neq 0$ for all $\vec{x} \in V$.

§2.5 #4.

$$u = x^2 + 3y^2 \qquad y - xz = 0$$

1) compute $\frac{\partial u}{\partial x}$ assuming that (x,y) independent, z dependent.

2) " " (x,z) " y "

3) " " (y,z) " x "

Write out as composite:

$$u = v^2 + 3w^2 \qquad v = x, \quad w = y$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial x}$$

$$= 2v \frac{\partial v}{\partial x} + 6w \frac{\partial w}{\partial x}$$

$$= 2x \frac{\partial v}{\partial x} + 6y \frac{\partial w}{\partial x}$$

in 1), this is $2x \frac{\partial x}{\partial x} + 6y \frac{\partial y}{\partial x} = 2x$

2), this is $2x \underbrace{\frac{\partial x}{\partial x}}_{=1} + 6y \underbrace{\frac{\partial y}{\partial x}}_{\neq 0,\ = z} = 2x + 6yz$

$$F(x,z,y) = y - xz = 0$$

$$\frac{\partial y}{\partial x} = -\frac{\partial F}{\partial x} \bigg/ \frac{\partial F}{\partial y} = -\frac{-z}{1} = z$$

3), this is $2x \frac{\partial x}{\partial x} + 6y \frac{\partial y}{\partial x}$

Taylor polynomials: (of order m).

$f: \overset{\text{open}}{U} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$. f is of class C^m. two forms:

Unsorted Taylor polynomial of order m.

$$f(\vec{a}+\vec{h}) \approx P_{f,\vec{a},m}(\vec{h}) = \sum_{|I|\leq m} \frac{\partial_I f(\vec{a})}{|I|!} \vec{h}_I$$

where $|I|$ is number of elements in I.

$h_I = h_{i_1} h_{i_2} \cdots h_{i_m}$ ↳ order of differentiation / multiplication

Sorted form:

$$f(\vec{a}+\vec{h}) \approx \sum_{|\alpha|\leq m} \frac{\partial^{\bar{\alpha}} f(\vec{a})}{\bar{\alpha}!} h^{\bar{\alpha}}$$

$\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ dimension.