Lec 2/3

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Mean Value theorem in $\mathbb{R}^n$

Theorem: Let L be a line segment in $\mathbb{R}^n$ with endpoints $\vec{a}, \vec{b}$. Suppose that $f: S \to \mathbb{R}$ is continuous, $L \subseteq S$, $L \setminus \{\vec{a}, \vec{b}\} \subseteq S^{int}$, f diffable on S^{int}.

Then $f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \cdot (\vec{b} - \vec{a})$ for some $\vec{c} \in L \setminus \{\vec{a}, \vec{b}\}$.

Proof: Let $\vec{\lambda}: [0,1] \to L \subseteq \mathbb{R}^n$ be given by $\vec{\lambda}(t) = (1-t)\vec{a} + t(\vec{b}) = \vec{a} + t(\vec{b} - \vec{a})$.

Then $D\vec{\lambda}(t) = \vec{\lambda}'(t) = \vec{b} - \vec{a}$ $\forall t \in (0,1)$.

Let $\psi: [0,1] \to \mathbb{R}$ be $f \circ \vec{\lambda}$. Then ψ is continuous on $[0,1]$ and

$$\psi'(t) = Df(\vec{\lambda}(t)) \cdot \vec{\lambda}'(t)$$
$$= \nabla f(\vec{\lambda}(t)) \cdot (\vec{b} - \vec{a})$$

ψ is diffable on $(0,1)$.

By 1-var MVT, $\psi(1) - \psi(0) = \psi'(t_0)(1-0)$ for some $t_0 \in (0,1)$.

$$f(\vec{\lambda}(1)) - f(\vec{\lambda}(0)) = \nabla f(\vec{\lambda}(t_0)) \cdot (\vec{b} - \vec{a})$$
$$f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \cdot (\vec{b} - \vec{a}) \quad \text{where } \vec{c} = \vec{\lambda}(t_0). \quad \blacksquare$$

Definition: $C \subseteq \mathbb{R}^n$ is convex if $\vec{a}, \vec{b} \in C \Rightarrow (1-t)\vec{a} + t\vec{b} \in C$ for $t \in [0,1]$

Corollary of MVT: If $\mathcal{U}$ ($\subseteq \mathbb{R}^n$) is (non-empty) open & convex ... $f: \mathcal{U} \to \mathbb{R}$ is diffable on $\mathcal{U}$, then the following two statements are equivalent:

(1) f is constant

(2) $\nabla f(\vec{x}) = \vec{0}$ $\forall \vec{x} \in \mathcal{U}$.

Proof: Let $\vec{a} \in \mathcal{U}$. then for any other point $\vec{b} \in \mathcal{U}$, $f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c})(\vec{b} - \vec{a})$

(1) ⇒ (2) obvious

but $\nabla f(\vec{c}) = 0$ so $f(\vec{b}) - f(\vec{a}) = 0$. $\blacksquare$

Proposition: In corollary, need only assume that $\mathcal{U}$ is connected.

Proof: Corollary + Lemma:

Lemma: If $\mathcal{U} \subseteq \mathbb{R}^n$ is connected & open, then for any $\vec{a}, \vec{b} \in \mathcal{U}$, there is a connectivity chain $\{C_i\}_{i=0}^{k}$ from $\vec{a}$ to $\vec{b}$ with each C_i an open ball.

$(\vec{a} \in C_0, \vec{b} \in C_k, \ C_{i-1} \cap C_i \neq \emptyset \ \forall i)$. (also, $C_i \subseteq \mathcal{U}$)

Proof: choose $\vec{a} \in \mathcal{U}$. let $S = \{\vec{x} \in \mathcal{U}$: there is a connectivity chain of open balls from $\vec{a}$ to $\vec{x}\}$.

Want to show $S = \mathcal{U}$.

(1) $\vec{a} \in S \Rightarrow S$ nonempty

(2) S is open: if $\vec{x} \in S$ and $\{C_i\}_{i=0}^k$ is the connectivity chain, then $C_k \subseteq S$ so $\vec{x}$ is an interior point.

(3) S is closed in $\mathcal{U}$ (not in $\mathbb{R}^n$ unless $\mathcal{U} = \mathbb{R}^n$):

Suppose $\vec{y} \in \partial S \cap \mathcal{U}$. Then for some $r > 0$, $B(r, \vec{y}) \subseteq \mathcal{U} \Rightarrow \exists x \in S \cap B(r, \vec{y})$

let $\{C_i\}_{i=0}^k$ be a conn. chain from $\vec{a}$ to $\vec{x}$. then $\{C_0, \dots, C_k, B(r, \vec{y})\}$ is a c.c. from $\vec{a}$ to $\vec{y}$ so $\vec{y} \in S$. ∎

Definition let $\mathcal{U} \subseteq \mathbb{R}^{n+1}$ be open, $F: \mathcal{U} \to \mathbb{R}$ diff-able. We say that $g: V \to (a,b) \subseteq \mathbb{R}$ is an implicit function defined by $F(\underset{\mathbb{R}^n}{\vec{x}}, \underset{\mathbb{R}}{y}) = 0$ if

$\hookrightarrow$ diff-able.

(1) $V \times (a,b) \subseteq \mathcal{U}$

(2) $\forall \vec{x} \in V$, $F(\vec{x}, g(x)) = 0$

(3) if $(\vec{x}, y) \in V \times (a,b)$ and $F(\vec{x}, y) = 0$ then $y = g(\vec{x})$.

Standard example: $F: \mathbb{R}^2 \to \mathbb{R}$

$F(x,y) = x^2 + y^2 - 1$

Then $g: (-1,1) \to (0,1)$ given by $g(x) = \sqrt{1-x^2}$

is an implicit function defined by $F(x,y) = 0$ on $(-1,1) \times (0,1)$.

Note: $h: (-1,1) \to (-1,0)$, $h(x) = -\sqrt{1-x^2}$ is another implicit function.

Assuming there is an implicit function as in the definition, we want to express $\dfrac{\partial g}{\partial x_i}$ in terms of $\partial_j F$ for $j = 1, \dots, n+1$ for $i = 1, \dots, n$.

Apply Chain rule: Define $\vec{\lambda}: \underset{\subseteq \mathbb{R}^n}{V} \to \underset{\subseteq \mathbb{R}^{n+1}}{\mathcal{U}}$ by $\vec{\lambda}(\vec{x}) = (\vec{x}, g(x))$

$$D\vec{\lambda}(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ \partial g & \partial g & & & \cdot \end{pmatrix}$$

$F \circ \vec{\lambda}(\vec{x}) = F(\vec{x}, g(\vec{x})) = 0$ by definition.

$\underset{1 \times n}{D(F \circ \vec{\lambda})(\vec{x})} = \underset{1 \times (n+1)}{DF(\vec{\lambda}(\vec{x}))} \cdot \underset{(n+1) \times n}{D\vec{\lambda}(x)} = 0$

$$\begin{pmatrix} 0 & 0 & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{x_2} & \cdots & \frac{\partial g}{x_n} \end{pmatrix}$$

$$\underset{1 \times n}{D(F \circ \vec{\lambda})(\vec{x})} = \underset{1 \times (n+1)}{DF(\vec{\lambda}(\vec{x}))} \cdot \underset{(n+1) \times n}{D\vec{\lambda}(x)} = 0$$

$$= \nabla F(\vec{\lambda}(\vec{x})) \cdot D\vec{\lambda}(x)$$

$$\Rightarrow 0 = (\partial_1 F, \partial_2 F, \ldots, \partial_{n+1} F) \cdot \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{pmatrix}$$

all $\partial_i F$ evaluated at $\vec{\lambda}(\vec{x})$

$$= \left(\partial_1 F + \partial_{n+1} F \frac{\partial g}{\partial x_1}, \partial_2 F + \partial_{n+1} F \frac{\partial g}{\partial x_2}, \ldots, \partial_n F + \partial_{n+1} F \frac{\partial g}{\partial x_n}\right)$$

$$= \vec{0} \implies \partial_i F(\vec{x}, g(\vec{x})) + \partial_{n+1} F(\vec{x}, g(\vec{x})) \frac{\partial g}{\partial x_i} = 0 \quad \text{for } i = 1, \ldots, n.$$

$$\implies \frac{\partial g}{\partial x_i} = \frac{-\partial_i F(\vec{x}, g(\vec{x}))}{\partial_{n+1} F(\vec{x}, g(\vec{x}))}$$

Necessary condition for such an implicit function to exist is that $\partial_{n+1} F(\vec{x}, g(\vec{x})) \neq 0$ for $\vec{x} \in V$. Later will show this is a sufficient condition.

(implicit function Theorem).